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On Brauer–Ostrowski and Brualdi sets

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A B S T R A C T

For the localization of the spectrum of the eigenvalues of a complex square matrix, the classical Geršgorin Theorem was extended by Ostrowski who used the *generalized geometric mean* of the row and column sums of the matrix. Ostrowski, and Brauer, extended the previous idea by using generalized geometric means of products of two row and column sums. Finally, by using the Graph Theory, Brualdi extended all of the previous ideas further by considering generalized geometric means of products of two or more than two row and column sums. These localization results can also provide classes of nonsingular matrices. Our main aim in this work is to exploit all the above known results and determine intervals for the parameter(s) α (α_k's) involved so that the localization of the spectrum in question as well as the determination of the associated class of nonsingular matrices are possible.

1. Introduction and preliminaries

The relatively recent book by Varga *"Geršgorin and His Circles"* [22] inspired many researchers in the area to exploit its background material and extend classical results

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(see, e.g., [9,7,23,8,12]). In [22], localization regions for the spectrum of the eigenvalues of a complex square matrix were obtained by using the results of Geršgorin's Theorem (see [10] or Theorem 1.11 of [20]), as well as those by Ostrowski [16], Brauer [2–4] and Brualdi [6]. In this work we determine intervals of the parameter(s) α (α_k's) involved in the aforementioned localization regions of spectra as well as the associated classes of nonsingular matrices for, mainly, irreducible matrices. The present theory extends previous results obtained in [8] and [12] by using the theory developed in the work by Brualdi [6], and the background material in Varga [22].

To begin with, consider the set of the first n positive integers denoted by $N := \{1, 2, \ldots, n\}$ and let $A \in \mathbb{C}^{n\times n}$, $n \geqslant 2$. For $A \in \mathbb{C}^{n\times n}$ let $r_i := \sum_{j\in N\setminus\{i\}} |a_{ij}|$, $\forall i \in N$ (ith row sum) and $c_j := \sum_{i\in N\setminus\{j\}} |a_{ij}|$, $\forall j \in N$ (jth column sum). $A \in \mathbb{C}^{n\times n}$ is *diagonally dominant (by rows)* (*DD* matrix) if and only if (*iff*) $|a_{ii}| \geqslant r_i$, $\forall i \in N$. $A \in \mathbb{C}^{n\times n}$ is *strictly diagonally dominant (by rows)* (*SDD* matrix) *iff* $|a_{ii}| > r_i$, $\forall i \in N$. $A \in \mathbb{C}^{n\times n}$ is *irreducibly diagonally dominant (by rows)* (*IDD* matrix) *iff* it is *irreducible* and *DD* with at least one inequality strict. (For more, see, e.g., Varga [20] and Berman and Plemmons [1].)

2. Review of known results

We begin with the well-known Lévy–Desplanques Theorem (see [14] and [11] or [15, p. 146], or Brualdi [6]).

Lemma 2.1. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, and let*

$$|a_{ii}| > r_i \quad (\forall i \in N). \tag{2.1}$$

Then A is nonsingular.

Geršgorin [10] presented the statement below from which (2.1) can also be recovered.

Lemma 2.2 *(Geršgorin Theorem). (See [10] or [20].) Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, then, if $\sigma(A)$ denotes the spectrum of the eigenvalues of A, there will hold*

$$\sigma(A) \subseteq \bigcup_{i=1}^{n} \mathcal{D}_i := \bigcup_{i=1}^{n} \{z \in \mathbb{C}\colon\ |z - a_{ii}| \leqslant r_i\}. \tag{2.2}$$

Lemma 2.1 covers the class of SDD matrices. Taussky [19] extended it to include the class of IDD matrices.

Lemma 2.3. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, be irreducible and let*

$$|a_{ii}| \geqslant r_i \quad (\forall i \in N), \tag{2.3}$$

with at least one inequality strict. Then A is nonsingular.

Ostrowski extended Lévy–Desplanques's Theorem 2.1 (see [16] or Theorem 1.16 of [22]) by using *generalized geometric means* of row and column sums.

Lemma 2.4. *Let* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, *and let*

$$|a_{ii}| > r_i^{\alpha} c_i^{1-\alpha} \quad (\forall i \in N) \tag{2.4}$$

hold for some $\alpha \in [0,1]$. *Then A is nonsingular. (Note: The convention* $0^0 = 1$ *will be used throughout this paper.)*

Ostrowski's Theorem has been extended in Taussky's spirit (see, e.g., [12]). Specifically:

Lemma 2.5. *Let* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, *be irreducible and let*

$$|a_{ii}| \geqslant r_i^{\alpha} c_i^{1-\alpha} \quad (\forall i \in N), \tag{2.5}$$

with at least one inequality strict, hold for some $\alpha \in [0,1]$. *Then A is nonsingular.*

A proposition due to Ostrowski [16] and rediscovered by Brauer [2] (see Theorem 2.1 of [22]) reads as follows:

Lemma 2.6. *Let* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, *and let*

$$|a_{ii}||a_{jj}| > r_i r_j \quad (\forall i \neq j \in N). \tag{2.6}$$

Then A is nonsingular.

Brualdi notes [6] that Brauer [4] had improved his oval inclusion region and had shown that the analogue of (2.3) for Lemma 2.6 also holds [3,4].

Ostrowski [17] generalized Theorem 2.6 as is given below.

Lemma 2.7. *Let* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, *and let*

$$|a_{ii}||a_{jj}| > r_i^{\alpha} c_i^{1-\alpha} r_j^{\alpha} c_j^{1-\alpha} \quad (\forall i \neq j \in N), \tag{2.7}$$

hold for some $\alpha \in [0,1]$. *Then A is nonsingular.*

Finally, Brualdi [6] extended all of the previous results using the *Graph Theory* (see, e.g., Harary [13]), where products of two or more than two row and column sums were considered. Two of his propositions are given below in a little different form from what they appear in [6].

Lemma 2.8. *(See Theorem 2.9 of Brualdi [6].) Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, be irreducible. Suppose*

$$\prod_{i\in\gamma_j} |a_{ii}| \geqslant \prod_{i\in\gamma_j} r_i \quad \bigl(\forall\gamma_j \in \mathcal{C}(A)\bigr) \tag{2.8}$$

hold for each cycle γ_j of the directed graph $\Gamma(A)$ of A, where no loops are considered and where $\mathcal{C}(A)$ is the set of all the cycles of $\Gamma(A)$. Let strict inequality hold in (2.8) for at least one cycle γ_j. Then A is nonsingular.

Note: It should be explained that the nodes $i_1, i_2, \ldots, i_p$ of a cycle γ_j are those for which $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{p-1} i_p} a_{i_p i_1} \neq 0$, and constitute the vertices of the cycle $\gamma_j = (i_1\, i_2\, i_3\, \ldots\, i_{p-1}\, i_p)$ where all i_k, $k = 1(1)p$, are distinct.

Lemma 2.9. *(See Corollary 2.13 of Brualdi [6].) Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, be weakly irreducible. Then for each α, with $\alpha \in [0,1]$, the eigenvalues of A lie in the union of the lemniscates*

$$\sigma(A) \subseteq \mathcal{B}(A) := \bigcup_{\gamma_j\in\mathcal{C}(A)} \left\{ z \in \mathbb{C}\colon \prod_{i\in\gamma_j} |z - a_{ii}| \leqslant \left(\prod_{i\in\gamma_j} r_i\right)^{\alpha} \left(\prod_{i\in\gamma_j} c_i\right)^{1-\alpha} \right\}. \tag{2.9}$$

Notes: The term "*weakly irreducible*", introduced by Brualdi [6], means that in the "*Frobenius normal form*" of A all diagonal blocks are of order $\geqslant 2$. From now on it will be assumed that a weakly irreducible matrix is given in its Frobenius normal form; it is reminded that the first who proposed an algorithm to determine the Frobenius normal form was Tarjan [18] while the most recent relevant algorithm can be found in [5]. Finally, whenever an equality (strict inequality) is met in (2.8), the corresponding lemniscate in (2.9) is open (closed) and vice versa.

Remark 2.1. It is pointed out that in the case of Lemmas 2.8 and 2.9, the quantities r_i (and c_i) are restricted to the row (and column) sums within the corresponding diagonal blocks; Varga [22] uses the notation $\widetilde{r}_i$ (and $\widetilde{c}_i$) for these quantities.

Note: For Lemmas 2.6–2.7 and also the one in [4], mentioned previously, statements analogous to Lemma 2.2 can be given as Brualdi did in [6]. Note that we do not deal with disks anymore, as in Lemmas 2.1–2.5, but with *Brauer–Ostrowski Cassini ovals* and *Brualdi lemniscates*; we use the term Brauer–Ostrowski Cassini ovals instead of *Brauer Cassini ovals*, as Varga does [22], because the latter term refers only to the case $\alpha = 1$ or $\alpha = 0$.

Our main objective in the present paper is to determine the intervals of the parameter(s) α (α_k's) of an irreducible (or of a weakly irreducible) matrix A to cover the case of Brualdi lemniscates of Lemmas 2.8–2.9 and, if possible, to extend the known theory.

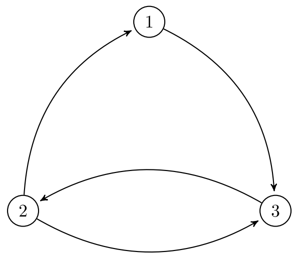

Fig. 1. Directed graph of matrices A_1 and A_2 of Example 1.

3. Brualdi lemniscates and sets

3.1. Introduction

We begin this section by pointing out that a direct implication of Lemmas 2.8 and 2.9 can be obtained as follows. Assuming that the Brualdi lemniscates in (2.9), and their union ("Brualdi set") contain the eigenvalue zero, an eigenvalue of the singular matrix $A \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, then by setting $z = 0$ in (2.9) and combining it with Lemma 2.8 for an irreducible matrix A we have that:

Theorem 3.1. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, be irreducible. Suppose*

$$\prod_{i\in\gamma_j} |a_{ii}| \geqslant \left(\prod_{i\in\gamma_j} r_i\right)^{\alpha} \left(\prod_{i\in\gamma_j} c_i\right)^{1-\alpha} \quad \bigl(\forall\gamma_j \in \mathcal{C}(A)\bigr) \tag{3.1}$$

hold for some $\alpha \in [0,1]$ and for each cycle γ_j of the directed graph $\Gamma(A)$ of A, with strict inequality for at least one cycle γ_j. Then A is nonsingular.

It should be pointed out that all the main statements so far that imply nonsingularity of a matrix give sufficient conditions only and **not** sufficient and necessary ones. The reason for this is clearly seen in the following example.

Example 1. Let the irreducible matrices A_1 and A_2 be as follows

$$A_1 = \begin{bmatrix} 3 & 0 & -3 \\ -3 & 3 & -1 \\ 0 & -\frac{1}{4} & \frac{1}{3} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 3 & 0 & -3 \\ -3 & 3 & 1 \\ 0 & \frac{1}{4} & \frac{1}{3} \end{bmatrix}. \tag{3.2}$$

As is seen A_1 and A_2 share the same graph of Fig. 1 which has the two cycles $\gamma_1 = (2\,3)$ and $\gamma_2 = (1\,3\,2)$. In addition, for the elements of the two matrices we have

$$|a_{11}| = 3, \qquad r_1 = 3, \qquad c_1 = 3,$$
$$|a_{22}| = 3, \qquad r_2 = 4, \qquad c_2 = \frac{1}{4},$$

$$|a_{33}| = \frac{1}{3}, \qquad r_3 = \frac{1}{4}, \qquad c_3 = 4,$$

which are the same for both matrices and satisfy relations of type (3.1). Specifically,

$$\prod_{i\in\gamma_1} |a_{ii}| = \left(\prod_{i\in\gamma_1} r_i\right)^{\alpha} \left(\prod_{i\in\gamma_1} c_i\right)^{1-\alpha} = 1,$$
$$\prod_{i\in\gamma_2} |a_{ii}| = \left(\prod_{i\in\gamma_2} r_i\right)^{\alpha} \left(\prod_{i\in\gamma_2} c_i\right)^{1-\alpha} = 3,$$

for any $\alpha \in [0,1]$. However, $\det(A_1) = 0$ while $\det(A_2) = 4.5$, which means that in case of equality in all relations of (3.1) the corresponding matrix may be singular or not.

We introduce the class of "*Brualdi matrices*" or simply "*$\mathcal{B}$-matrices*" as follows.

Definition 3.1. An irreducible matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, satisfying the assumptions of Theorem 3.1 for some $\alpha \in [0,1]$ will be called a $\mathcal{B}$-matrix. (*Note:* By Theorem 3.1 a $\mathcal{B}$-matrix is nonsingular.)

3.2. Determination of α

To find possible ranges for α, if they exist, and check if the assumptions of Definition 3.1 hold for a certain irreducible matrix A, we write the desired inequalities in (3.1) as

$$\frac{\prod_{i\in\gamma_j} |a_{ii}|}{\prod_{i\in\gamma_j} c_i} \geqslant \left(\frac{\prod_{i\in\gamma_j} r_i}{\prod_{i\in\gamma_j} c_i}\right)^{\alpha} \quad \bigl(\forall\gamma_j \in \mathcal{C}(A)\bigr). \tag{3.3}$$

To simplify the notation, we set

$$|A_{i\in\gamma_j}| := \prod_{i\in\gamma_j} |a_{ii}|, \qquad R_{i\in\gamma_j} := \prod_{i\in\gamma_j} r_i, \qquad C_{i\in\gamma_j} := \prod_{i\in\gamma_j} c_i, \tag{3.4}$$

and write (3.3), equivalently, as

$$\frac{|A_{i\in\gamma_j}|}{C_{i\in\gamma_j}} \geqslant \left(\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}}\right)^{\alpha} \quad \bigl(\forall\gamma_j \in \mathcal{C}(A)\bigr). \tag{3.5}$$

Considering all possible orderings of the quantities $|A_{i\in\gamma_j}|$, $R_{i\in\gamma_j}$, $C_{i\in\gamma_j}$, we can construct a table. For example if $R_{i\in\gamma_j} > |A_{i\in\gamma_j}| > C_{i\in\gamma_j}$, then both the left and the right sides of (3.5) are greater than 1 and, obviously, for all α's in the interval shown on the left

Table 1
Values of α for which $\frac{|A_{i\in\gamma_j}|}{C_{i\in\gamma_j}} \geqslant (\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}})^\alpha$ holds as a strict inequality or as an equality.

Cases		Values of α for which relation $\frac{\lvert A_{i\in\gamma_j}\rvert}{C_{i\in\gamma_j}} \geqslant (\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}})^\alpha$ holds as: A strict inequality	An equality
(i)	$\lvert A_{i\in\gamma_j}\rvert = R_{i\in\gamma_j} = C_{i\in\gamma_j}$	–	$\alpha \in [0,1]$
(ii)	$\lvert A_{i\in\gamma_j}\rvert > \max\{R_{i\in\gamma_j}, C_{i\in\gamma_j}\}$	$[0,1]$	–
(iii)	$R_{i\in\gamma_j} > \lvert A_{i\in\gamma_j}\rvert = C_{i\in\gamma_j}$	–	$\alpha = 0$
(iv)	$R_{i\in\gamma_j} > \lvert A_{i\in\gamma_j}\rvert > C_{i\in\gamma_j}$	$\alpha \in \Big[0, \frac{\log(\frac{\lvert A_{i\in\gamma_j}\rvert}{C_{i\in\gamma_j}})}{\log(\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}})} (< 1)\Big)$	$\alpha = \frac{\log(\frac{\lvert A_{i\in\gamma_j}\rvert}{C_{i\in\gamma_j}})}{\log(\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}})}$
(v)	$\lvert A_{i\in\gamma_j}\rvert = R_{i\in\gamma_j} > C_{i\in\gamma_j}$	$\alpha \in [0,1)$	$\alpha = 1$
(vi)	$C_{i\in\gamma_j} > \lvert A_{i\in\gamma_j}\rvert = R_{i\in\gamma_j}$	–	$\alpha = 1$
(vii)	$C_{i\in\gamma_j} > \lvert A_{i\in\gamma_j}\rvert > R_{i\in\gamma_j}$	$\alpha \in \Big((0 <) \frac{\log(\frac{\lvert A_{i\in\gamma_j}\rvert}{C_{i\in\gamma_j}})}{\log(\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}})}, 1\Big]$	$\alpha = \frac{\log(\frac{\lvert A_{i\in\gamma_j}\rvert}{C_{i\in\gamma_j}})}{\log(\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}})}$
(viii)	$\lvert A_{i\in\gamma_j}\rvert = C_{i\in\gamma_j} > R_{i\in\gamma_j}$	$\alpha \in (0,1]$	$\alpha = 0$

below, (3.5) holds as a strict inequality while for the value of α on the right, (3.5) holds as an equality

$$\alpha \in \left[0, \frac{\log(\frac{|A_{i\in\gamma_j}|}{C_{i\in\gamma_j}})}{\log(\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}})}\right), \qquad \alpha = \frac{\log(\frac{|A_{i\in\gamma_j}|}{C_{i\in\gamma_j}})}{\log(\frac{R_{i\in\gamma_j}}{C_{i\in\gamma_j}})}. \tag{3.6}$$

Therefore, if for each $\gamma_j \in \mathcal{C}(A)$ there exists an interval of α for which (3.5) holds and the intersection of all these intervals is a nonempty set and does **not** come exclusively from equalities in (3.5), then this set will contain all values of α for which relations (3.5) will hold simultaneously, and the conclusion will be that A belongs to the class of $\mathcal{B}$-matrices.

Note that relations (3.5) are of the same nature as relations $\frac{|a_{ii}|}{c_i} \geqslant (\frac{r_i}{c_i})^\alpha$ $(\forall i \in N)$ considered in [12]. The latter relations led to the construction of a table. So, a similar analysis, which is omitted here, leads to the construction of an analogous table, Table 1.

Based on Definition 3.1, the preceding discussion and Table 1, we can state and prove the following theorem.

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, be irreducible and that for each cycle $\gamma_j \in \mathcal{C}(A)$, $j = 1(1)k$, there exists an interval of α, let it be $\mathcal{S}_j := \{\alpha_{l_j}, \alpha_{r_j}\}$, for which a case of Table 1 applies; otherwise $\mathcal{S}_j = \emptyset$. Then, A belongs to the class of $\mathcal{B}$-matrices iff $\alpha \in \mathcal{S} := \bigcap_{j=1}^k \mathcal{S}_j$, where $\mathcal{S} \neq \emptyset$ and $\mathcal{S}$ does* **not** *come exclusively from equalities of Table 1. (Note: The symbol "{" denotes either "(" for an open left or "[" for a closed left interval, whichever applies; similarly, "}" denotes a right open or closed interval.)*

Proof. The proof will be given for $k = 2$ since it is easily extended to any $k > 2$ by induction. Let $\gamma_j \in \mathcal{C}(A)$, $j = 1, 2$, be the only cycles of the directed graph $\Gamma(A)$ of the irreducible matrix A. Suppose that the orderings of the quantities in (3.4) for each $\gamma_j \in \mathcal{C}(A)$ belong to Cases of Table 1 and let $\mathcal{S}_j = \{\alpha_{l_j}, \alpha_{r_j}\}$, $j = 1, 2$, be the

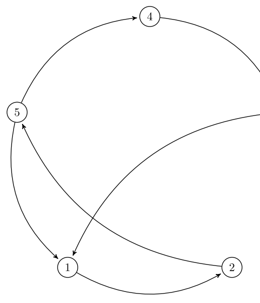

Fig. 2. Directed graph of matrix A of Example 2.

corresponding intervals for α. Let that $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ and $\mathcal{S}$ does not come exclusively from equalities in Table 1. Then, for any $\alpha \in \mathcal{S}$ the orderings of the quantities in (3.4) for each $\gamma_j \in \mathcal{C}(A)$ will satisfy relations of type (3.5) or, equivalently, of (3.1), where at least one of them must be strict since $\mathcal{S}$ does not come exclusively from equalities of Table 1. Consequently, for all $\alpha \in \mathcal{S}$ the matrix A is a $\mathcal{B}$-matrix by Definition 3.1.

Conversely, let A be a $\mathcal{B}$-matrix. Then, for each cycle $\gamma_j \in \mathcal{C}(A)$, $j = 1, 2$, relations of type (3.1) are satisfied for some α's in intervals $\mathcal{S}_j = \{\alpha_{l_j}, \alpha_{r_j}\}$, $j = 1, 2$. Since A is a $\mathcal{B}$-matrix, there must be α's belonging to $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2$ satisfying (3.1). Hence, $\mathcal{S}$ is not empty and does not come exclusively from equalities in Table 1 since at least one of the associated inequalities (3.5) is strict due to the $\mathcal{B}$-matrix character of A. □

Note: It is pointed out that in Example 1, $\mathcal{S}$ comes exclusively from equalities of Table 1, Case (i), and so neither A_1 nor A_2 is a $\mathcal{B}$-matrix.

Example 2. Suppose $A \in \mathbb{C}^{5\times 5}$ is irreducible and its directed graph $\Gamma(A)$ contains only the two cycles $\gamma_1 = (1\,2\,5\,4\,3)$, satisfying the ordering of Case (iv), and $\gamma_2 = (1\,2\,5)$, satisfying that of Case (vii) (see Fig. 2). Then, from Table 1 we have that

$$\begin{aligned} \alpha \in \mathcal{S}_1 = [\alpha_{l_1}, \alpha_{r_1}] &\equiv \left[0, \frac{\log(\frac{|A_{i\in\gamma_1}|}{C_{i\in\gamma_1}})}{\log(\frac{R_{i\in\gamma_1}}{C_{i\in\gamma_1}})}\right], \\ \alpha \in \mathcal{S}_2 = [\alpha_{l_2}, \alpha_{r_2}] &\equiv \left[\frac{\log(\frac{|A_{i\in\gamma_2}|}{C_{i\in\gamma_2}})}{\log(\frac{R_{i\in\gamma_2}}{C_{i\in\gamma_2}})}, 1\right]. \end{aligned} \tag{3.7}$$

If $\alpha_{l_2} < \alpha_{r_1}$, then A is a $\mathcal{B}$-matrix for all $\alpha \in \mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 = [\alpha_{l_2}, \alpha_{r_1}]$ and

$$\prod_{i\in\gamma_j} |a_{ii}| \geqslant \Bigl(\prod_{i\in\gamma_j} r_i\Bigr)^{\alpha} \Bigl(\prod_{i\in\gamma_j} c_i\Bigr)^{1-\alpha}, \quad j = 1, 2, \tag{3.8}$$

hold. Note that for $\alpha \in (\alpha_{l_2}, \alpha_{r_1})$ both relations in (3.8) are strict while for $\alpha = \alpha_{r_1}$ and $\alpha = \alpha_{l_2}$ it is

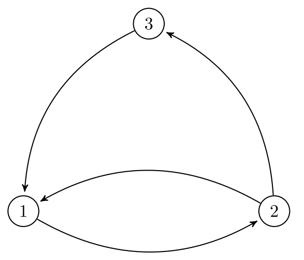

Fig. 3. Directed graph of matrix A of Example 3.

$$\prod_{i\in\gamma_1}|a_{ii}| = \Big(\prod_{i\in\gamma_1} r_i\Big)^{\alpha}\Big(\prod_{i\in\gamma_1} c_i\Big)^{1-\alpha} \quad \text{and}$$

$$\prod_{i\in\gamma_2}|a_{ii}| = \Big(\prod_{i\in\gamma_2} r_i\Big)^{\alpha}\Big(\prod_{i\in\gamma_2} c_i\Big)^{1-\alpha}, \tag{3.9}$$

respectively, meaning that the corresponding relation in (3.8) will be an equality.

If $\alpha_{l_2} = \alpha_{r_1} =: \alpha$, then there is a unique value of α (the ratio of the two logarithms in either of (3.7)) for which (3.8) hold as equalities and by Theorem 3.2 A is **not** a $\mathcal{B}$-matrix and so **no** conclusion as regards the nonsingularity of A can be drawn.

Finally, if $\alpha_{r_1} < \alpha_{l_2}$, A is not a $\mathcal{B}$-matrix and again, as above, **no** conclusion can be drawn.

In the sequel we give two specific examples where the classical Ostrowski Theorem does **not** apply for any value of α while Theorem 3.1 **does** for all α's in an interval to be determined by Theorem 3.2.

Example 3. Let the irreducible matrix be as follows

$$A = \begin{bmatrix} e^{\imath\theta_{11}} & 1.05e^{\imath\theta_{12}} & 0 \\ 0.6e^{\imath\theta_{21}} & e^{\imath\theta_{22}} & 0.3e^{\imath\theta_{23}} \\ 1.1e^{\imath\theta_{31}} & 0 & e^{\imath\theta_{33}} \end{bmatrix}, \tag{3.10}$$

where $\imath$ is the imaginary unit and θ_{ij}, $i, j \in \{1, 2, 3\}$, are any real numbers. As is seen, $|a_{11}| = 1 < \min\{r_1, c_1\} = 1.05$. Therefore, none of the two Lemmas 2.4 and 2.5 can be applied as this is known from Table 1 in [12], which is analogous to Table 1 presented before. However, Theorem 3.2 can be applied as we will see in the sequel.

The directed graph $\Gamma(A)$ is seen in Fig. 3. Obviously, there are two cycles $\gamma_1 = (1\,2)$ and $\gamma_2 = (1\,2\,3)$. To find the Brualdi set it suffices to consider the two lemniscates corresponding to the two cycles. Thus we have

$$\begin{aligned} |A_{12}| &= 1, & R_{12} &= 0.945, & C_{12} &= 1.785 &\Longrightarrow&\quad \text{Case (vii)}, \\ |A_{123}| &= 1, & R_{123} &= 1.0395, & C_{123} &= 0.5355 &\Longrightarrow&\quad \text{Case (iv)}. \end{aligned} \tag{3.11}$$

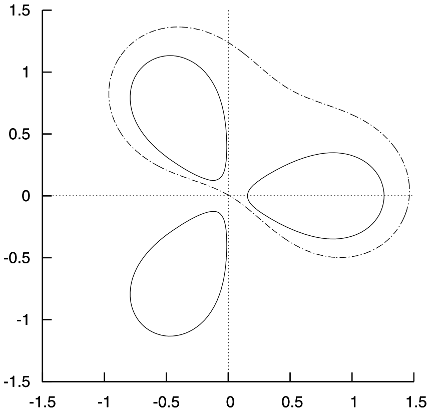

Fig. 4. The Brualdi set $\mathcal{B}(A)$ of Example 3 constitutes of the Brauer–Ostrowski Cassini oval (dashed line) and the component of the Brualdi lemniscate (solid line) located in the third quadrant.

From Table 1 it is readily seen that the above conclusions hold for any value of

$$\alpha \in \left[\frac{\log(\frac{1}{1.785})}{\log(\frac{0.945}{1.785})}, \frac{\log(\frac{1}{0.5355})}{\log(\frac{1.0395}{0.5355})}\right] \equiv [0.9111, 0.9416], \tag{3.12}$$

to four decimal places, and both relations (3.13) below hold as strict inequalities for all $\alpha \in (0.9111, 0.9416)$, while for $\alpha = 0.9111$ and for $\alpha = 0.9416$ the first and the second relations below hold as equalities, respectively. More specifically,

$$|A_{12}| \geqslant (R_{12})^{\alpha}(C_{12})^{1-\alpha}, \qquad |A_{123}| \geqslant (R_{123})^{\alpha}(C_{123})^{1-\alpha}. \tag{3.13}$$

For all the aforementioned values of α, A is a $\mathcal{B}$-matrix and, therefore, nonsingular. Moreover, for the Brauer–Ostrowski Cassini oval and the Brualdi lemniscate we have that

$$\begin{aligned} \mathcal{K}_{(1\,2)}(A) &:= \left\{z \in \mathbb{C}\colon \left|z - e^{\imath\theta_{11}}\right|\left|z - e^{\imath\theta_{22}}\right| \leqslant (0.945)^{\alpha}(1.785)^{1-\alpha}\right\}, \\ \mathcal{B}_{(1\,2\,3)}(A) &:= \left\{z \in \mathbb{C}\colon \left|z - e^{\imath\theta_{11}}\right|\left|z - e^{\imath\theta_{22}}\right|\left|z - e^{\imath\theta_{33}}\right| \leqslant (1.0395)^{\alpha}(0.5355)^{1-\alpha}\right\}. \end{aligned} \tag{3.14}$$

It is pointed out that for α taking the values $\alpha = 0.9111$ and $\alpha = 0.9416$, $\mathcal{K}_{(1\,2)}(A)$ and $\mathcal{B}_{(1\,2\,3)}(A)$, respectively, are open. Finally, for the spectrum of A it is

$$\sigma(A) \subset \mathcal{B}(A) = \mathcal{K}_{(1\,2)} \cup \mathcal{B}_{(1\,2\,3)},$$

where $\mathcal{B}(A)$ is the Brualdi set. In Fig. 4 the Brauer–Ostrowski Cassini oval $\mathcal{K}_{(1\,2)}(A)$ and the Brualdi lemniscate $\mathcal{B}_{(1\,2\,3)}(A)$ as well as the Brualdi set $\mathcal{B}(A)$, in case $\theta_{11} = 0$, $\theta_{22} = \frac{2\pi}{3}$, $\theta_{33} = \frac{4\pi}{3}$ and $\alpha = 0.93$, are depicted. Note that the Brualdi set is the union of the disjoint sets $\mathcal{K}_{(1\,2)}$ and the component of $B_{(1\,2\,3)}(A)$ in the third quadrant. Also, note that both components of $\mathcal{B}(A)$ are closed.

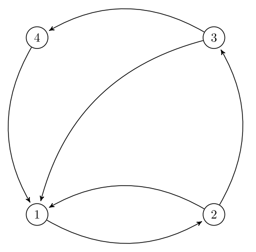

Fig. 5. Directed graph of matrix A of Example 3.

Example 4. Let the irreducible matrix be as follows

$$A = \begin{bmatrix} 1 & 0.8 & 0 & 0 \\ 0.25 & 1 & 0.95 & 0 \\ 0.25 & 0 & 1 & 0.5 \\ 1.4 & 0 & 0 & 1 \end{bmatrix}.$$

For the given matrix we have

$$\begin{aligned} a_{ii} &= 1, \qquad i = 1(1)4, \\ r_1 = 0.8, \quad r_2 &= 1.2, \quad r_3 = 0.75, \quad r_4 = 1.4, \\ c_1 = 1.9, \quad c_2 &= 0.8, \quad c_3 = 0.95, \quad c_4 = 0.5. \end{aligned}$$

First, we examine if either Lemma 2.4 or 2.5 applies. From the orderings

$$c_1 > |a_{11}| > r_1, \qquad r_2 > |a_{22}| > c_2, \qquad |a_{33}| > c_3 > r_3, \qquad r_4 > |a_{44}| > c_4,$$

and the use of Table 1 in [12], or even Table 1 we find that

$$\begin{aligned} 1.9 > 1 > 0.8 \quad &\Longrightarrow \quad \alpha \in \left[\frac{\log(\frac{1}{1.9})}{\log(\frac{0.8}{1.9})}, 1\right] \bigl(\equiv [0.7420, 1]\bigr), \\ 1.2 > 1 > 0.8 \quad &\Longrightarrow \quad \alpha \in \left[0, \frac{\log(\frac{1}{0.8})}{\log(\frac{1.2}{0.8})}\right] \bigl(\equiv [0, 0.5503]\bigr), \\ 1 > 0.95 > 0.75 \quad &\Longrightarrow \quad \alpha \in [0, 1], \\ 1.4 > 1 > 0.5 \quad &\Longrightarrow \quad \alpha \in \left[0, \frac{\log(\frac{1}{0.5})}{\log(\frac{1.4}{0.5})}\right] \bigl(\equiv [0, 0.6732]\bigr). \end{aligned} \tag{3.15}$$

As is seen above there is **no** common value of α belonging to all four intervals found in (3.15). So, the matrix A is **not** a $\mathcal{B}$-matrix and no conclusion regarding the nonsingularity of A can be drawn.

However, we can check by a simple directed graph (see Fig. 5) that there are three cycles in it; specifically, $\gamma_1 = (1\,2)$, $\gamma_2 = (1\,2\,3)$, $\gamma_3 = (1\,2\,3\,4)$.

Considering the cycles $\gamma_1 = (1\,2)$, $\gamma_2 = (1\,2\,3)$ and $\gamma_3 = (1\,2\,3\,4)$ and using Table 1 we find the three intervals for α below

$$[0.9112, 1], \quad [0.5280, 1], \quad [0, 0.9761],$$

respectively. This time there is a common interval for $\alpha \in [0.9112, 0.9761]$. Therefore,

$$\prod_{\gamma_j} |a_{ii}| \geqslant \left(\prod_{\gamma_j} r_i\right)^{\alpha} \left(\prod_{\gamma_j} c_i\right)^{1-\alpha}, \quad j = 1, 2, 3, \ \forall \alpha \in [0.9112, 0.9761], \tag{3.16}$$

the matrix A is a $\mathcal{B}$-matrix and, therefore, nonsingular. Note that for all $\alpha \in (0.9112, 0.9761)$ the three relations in (3.16) are strict inequalities, while for $\alpha = 0.9112$, the relation for γ_1 is equality and for $\alpha = 0.9761$, the one for γ_3 is equality; for these two extreme values for α the other two relations remain strict.

For the Brualdi set $\mathcal{B}_{\gamma_j \in \mathcal{C}}(A)$ that contains the spectrum $\sigma(A)$ we have

$$\begin{aligned}
\mathcal{B}_{\gamma_1} &:= \left\{z \in \mathbb{C}:\ |z - 1|^2 \leqslant (0.8 \times 1.2)^{\alpha} (1.9 \times 0.8)^{1-\alpha}\right\}, \\
\mathcal{B}_{\gamma_2} &:= \left\{z \in \mathbb{C}:\ |z - 1|^3 \leqslant (0.8 \times 1.2 \times 0.75)^{\alpha} (1.9 \times 0.8 \times 0.95)^{1-\alpha}\right\}, \\
\mathcal{B}_{\gamma_3} &:= \left\{z \in \mathbb{C}:\ |z - 1|^4 \leqslant (0.8 \times 1.2 \times 0.75 \times 1.4)^{\alpha} (1.9 \times 0.8 \times 0.95 \times 0.5)^{1-\alpha}\right\},
\end{aligned} \tag{3.17}$$

and, therefore,

$$\sigma(A) \subset \mathcal{B}_{\gamma_j \in \mathcal{C}}(A) := \mathcal{B}_{\gamma_1} \cup \mathcal{B}_{\gamma_2} \cup \mathcal{B}_{\gamma_3}.$$

Note that for $\alpha = 0.9112$ and $\alpha = 0.9761$, $\mathcal{B}_{\gamma_1}$ and $\mathcal{B}_{\gamma_3}$ will be open Cassini ovals, respectively.

From (3.17) it is clear that the Cassini ovals are concentric disks. Specifically, (3.17) become

$$\begin{aligned}
\mathcal{B}_{\gamma_1} &:= \left\{z \in \mathbb{C}:\ |z - 1| \leqslant \left((0.8 \times 1.2)^{\alpha} (1.9 \times 0.8)^{1-\alpha}\right)^{\frac{1}{2}} =: f_1(\alpha)\right\}, \\
\mathcal{B}_{\gamma_2} &:= \left\{z \in \mathbb{C}:\ |z - 1| \leqslant \left((0.8 \times 1.2 \times 0.75)^{\alpha} (1.9 \times 0.8 \times 0.95)^{1-\alpha}\right)^{\frac{1}{3}} =: f_2(\alpha)\right\}, \\
\mathcal{B}_{\gamma_3} &:= \left\{z \in \mathbb{C}:\ |z - 1| \leqslant \left((0.8 \times 1.2 \times 0.75 \times 1.4)^{\alpha} (1.9 \times 0.8 \times 0.95 \times 0.5)^{1-\alpha}\right)^{\frac{1}{4}} \right. \\
&\qquad \left. =: f_3(\alpha)\right\}.
\end{aligned} \tag{3.18}$$

It is found that in the interval for $\alpha \in [0.9112, 0.9761]$, $f_1(\alpha)$, $f_2(\alpha)$ are strictly decreasing functions of α, while $f_3(\alpha)$ is strictly increasing. Also,

$$\begin{array}{ccc}
f_1(0.9761) = 0.9852, & f_2(0.9761) = 0.9013, & f_3(0.9761) = 1.0000, \\
f_1(0.9112) = 1.000, & f_2(0.9112) = 0.9149, & f_3(0.9112) = 0.9946.
\end{array}$$

The function $f(\alpha) = f_1(\alpha) - f_3(\alpha)$ is a continuous and strictly decreasing function of α in the above interval with $f(0.9761) = -0.0148 < 0$ and $f(0.9112) = 0.0054 > 0$. Consequently, there exists a unique value of $\alpha \in (0.9112, 0.9761)$, let it be α^*, such that $f(\alpha^*) = 0$. It is found that $\alpha^* = \frac{\log(\frac{c_3 c_4}{c_1 c_2})}{\log(\frac{r_1 r_2 c_3 c_4}{c_1 c_2 r_3 r_4})} = 0.9285$. As a consequence we have that the disk $\mathcal{B}_{\gamma_2}$ is always the inner disk, the disk $\mathcal{B}_{\gamma_1}$ will be the outer disk for $\alpha \in [0, 9112, 0.9285)$, while for $\alpha \in (0.9285, 0.9761]$, $\mathcal{B}_{\gamma_3}$ will be the outer one. For $\alpha = a^* = 0.9285$, $\mathcal{B}_{\gamma_3} \equiv \mathcal{B}_{\gamma_1}$.

The definition for "*Brualdi matrices*" ("*$\mathcal{B}$-matrices*"), Definition 3.1, can be extended to "*generalized Brualdi matrices*" ("*generalized $\mathcal{B}$-matrices*") in case we are dealing with weakly irreducible matrices.

Definition 3.2. Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geqslant 4$, be weakly irreducible and let A_{kk}, $k = 1(1)p$, be the diagonal blocks of its Frobenius normal form of respective orders $n_k \geqslant 2$, $k = 1(1)p$, $p \geqslant 2$, with $\sum_{k=1}^{p} n_k = n$. Then, A is called a "*generalized Brualdi matrix*" (*generalized $\mathcal{B}$-matrix*) *iff* each A_{kk}, $k = 1(1)p$, is a $\mathcal{B}$-matrix.

Remark 3.1. In case $A \in \mathbb{C}^{n\times n}$ is a *generalized $\mathcal{B}$-matrix*, let $\gamma_{j,k} \in \mathcal{C}_k$, $k = 1(1)p$, be the cycles associated with each A_{kk} and $\mathcal{S}_k$, $k = 1(1)p$, be the interval $\mathcal{S}$ defined in Theorem 3.2 for each A_{kk}. Then, there exist p-tuples of α, $(\alpha_1, \alpha_2, \ldots, \alpha_p)$, with $\alpha_k \in \mathcal{S}_k$, $k = 1(1)p$, such that

$$\prod_{i\in\gamma_{j,k}} |a_{ii}| \geqslant \left(\prod_{i\in\gamma_{j,k}} r_i\right)^{\alpha_k} \left(\prod_{i\in\gamma_{j,k}} c_i\right)^{1-\alpha_k} \quad \bigl(\forall \gamma_{j,k} \in \mathcal{C}_k, \quad \forall \mathcal{C}_k \in \mathcal{C}(A)\bigr) \tag{3.19}$$

hold, with strict inequality for at least one $\gamma_{j,k}$ in each $\mathcal{C}_k$. In addition, if $\mathcal{S} = \bigcap_{k=1}^{p} \mathcal{S}_k \neq \emptyset$, then besides the aforementioned choice of the p-tuples of α's one may choose a single value for $\alpha \in \mathcal{S}$. In such a case relations (3.19) hold with $\alpha_k = \alpha$, $k = 1(1)p$.

A trivial extension of the Brualdi–Varga Theorem 2.10 of [22] for irreducible matrices $A \in \mathbb{C}^{n\times n}$, $n \geqslant 3$, is as follows:

Theorem 3.3. *Under the assumptions of Lemma 2.9, except that the matrix $A \in \mathbb{C}^{n\times n}$, $n \geqslant 3$, is irreducible, let $V(\gamma_i)$ denote the set of the vertices of the cycle $\gamma_i \in \mathcal{C}(A)$, $i = 1(1)s$, $s \geqslant 2$, such that*

(i) $V(\gamma_1) = \bigcup_{j=2}^{s} V(\gamma_j)$, *and*
(ii) *there is a positive integer m such that each vertex from γ_1 appears exactly m times in $\bigcup_{j=2}^{s} V(\gamma_j)$.* (3.20)

Then, the Brualdi lemniscate $\mathcal{B}_{\gamma_1}$ can be removed from the Brualdi set $\mathcal{B}(A)$.

Proof. The proof follows exactly the same steps as those of Theorem 2.10 in [22], except that quantities of the type r_i in the latter are replaced by $r_i^\alpha c_i^{1-\alpha}$ in the former. So, the proof is omitted and the reader is referred to the original one in [22]. □

Corollary 3.1. *If the matrix A in the above theorem is totally dense then the Brualdi set coincides with the Brauer–Ostrowksi set.*

The following example covers the last four statements.

Example 5. Let $A \in \mathbb{C}^{7\times 7}$ be the weakly irreducible matrix given in its Frobenius normal form below

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{11} = \begin{bmatrix} 1 & 0.75 & 0 & 0 \\ 0.5 & 1.5 & 0.95 & 0 \\ 0.5 & 0 & 2 & 0.5 \\ 1.4 & 0 & 0 & 0.5 \end{bmatrix}, \qquad A_{12} \in \mathbb{C}^{4\times 3} \text{ any},$$

$$A_{21} = O_{3,4}, \qquad A_{22} = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.7 & 1.5 & 0.9 \\ 0.5 & \textcircled{x} & 1.8 \end{bmatrix},$$

and where x takes, in turn, the values $x = 0.7, 0.56, 0.49$. The objective in this example is to find whether for any of the above values of x the matrix A is a *generalized $\mathcal{B}$-matrix*.

In the first diagonal block A_{11} there are three cycles $\gamma_{1,1} = (1\,2)$, $\gamma_{2,1} = (1\,2\,3)$, $\gamma_{3,1} = (1\,2\,3\,4)$, with $\gamma_{1,1}, \gamma_{2,1}, \gamma_{3,1} \in \mathcal{C}_1$, and its graph is that of Fig. 5. For the various quantities needed we have

$$\begin{array}{llll} |a_{11}| = 1, & |a_{22}| = 1.5, & |a_{33}| = 2, & |a_{44}| = 0.5, \\ r_1 = 0.75, & r_2 = 1.45, & r_3 = 1, & r_4 = 1.4, \\ c_1 = 2.4, & c_2 = 0.75, & c_3 = 0.95, & c_4 = 0.5. \end{array}$$

For each of the three cycles using the values for $|a_{ii}|$'s, r_i's and c_i's and Table 1 we can find an interval for α. Specifically,

$$\mathcal{S}_{1,1} = [0.3618, 1], \qquad \mathcal{S}_{2,1} = [0, 1], \qquad \mathcal{S}_{3,1} = [0, 0.9742].$$

Hence, a common interval for α exists which is

$$\mathcal{S}_1 = \mathcal{S}_{1,1} \cap \mathcal{S}_{2,1} \cap \mathcal{S}_{3,1} = [0.3618, 0.9742],$$

implying that A_{11} is a $\mathcal{B}$-matrix.

The second diagonal submatrix A_{22} is totally dense and by Corollary 3.1 we have to consider only the three cycles $\gamma_{1,2} = (1\,2)$, $\gamma_{2,2} = (1\,3)$, $\gamma_{3,2} = (2\,3)$, $\gamma_{1,2}, \gamma_{2,2}, \gamma_{3,2} \in \mathcal{C}_2$.

For $\underline{x = 0.7}$, using the values for $|a_{ii}|$'s, r_i's and c_i's as well as Table 1, we can find the three intervals for α for each cycle. These are given below

$$\mathcal{S}_{1,2} = [0, 0.1110], \qquad \mathcal{S}_{2,2} = [0.4666, 1], \qquad \mathcal{S}_{3,2} = [0, 1].$$

As is seen $\mathcal{S}_2 = \mathcal{S}_{1,2} \cap \mathcal{S}_{2,2} \cap \mathcal{S}_{3,2} = \emptyset$. Hence, there is **no** common interval for α. Consequently, A_{22} is not a $\mathcal{B}$-matrix and, so, A is not a generalized $\mathcal{B}$-matrix.

For $\underline{x = 0.56}$, working in the same way, we find for the three cycles the intervals for α below

$$\mathcal{S}_{1,2} = [0, 0.3353], \qquad \mathcal{S}_{2,2} = [0.3190, 1], \qquad \mathcal{S}_{3,2} = [0, 1],$$

respectively. This time we have $\mathcal{S}_2 = \mathcal{S}_{1,2} \cap \mathcal{S}_{2,2} \cap \mathcal{S}_{3,2} = [0.3190, 0.3353]$. This means that A_{22} is a $\mathcal{B}$-matrix and A is a generalized $\mathcal{B}$-matrix. Since $\mathcal{S}_2 = \mathcal{S}_{1,2} \cap \mathcal{S}_{2,2} \cap \mathcal{S}_{3,2} \neq \emptyset$ we can always choose any $\alpha = \alpha_1 \in \mathcal{S}_1$ and any $\alpha = \alpha_2 \in \mathcal{S}_2$ and according to Theorem 3.1 we will have relations

$$\prod_{i \in \gamma_{j,k}} |a_{ii}| \geqslant \Bigg(\prod_{i \in \gamma_{j,k}} r_i \Bigg)^{\alpha_k} \Bigg(\prod_{i \in \gamma_{j,k}} c_i \Bigg)^{1-\alpha_k} \quad \big(\forall \gamma_{j,k} \in \mathcal{C}_k,\ \forall \mathcal{C}_k \in \mathcal{C}(A)\big), \tag{3.21}$$

where $\gamma_{j,1} \in \mathcal{C}_1$, $j = 1, 2, 3$, and $\gamma_{j,2} \in \mathcal{C}_2$, $j = 1, 2, 3$, hold. Note that all the relations in (3.21) are strict except in the following cases where they are equalities: (i) $a_1 = 0.3618$ for $\gamma_{1,1}$, (ii) $a_1 = 0.9742$ for $\gamma_{3,1}$, (iii) $\alpha_2 = 0.3190$ for $\gamma_{2,2}$ and (iv) $\alpha_2 = 0.3553$ for $\gamma_{1,2}$; also, in the corresponding relations in (3.21) for any combination of α's from (i) or (ii) with (iii) or (iv).

For $\underline{x = 0.49}$, working analogously, we find for the three cycles the intervals

$$\mathcal{S}_{1,2} = [0, 0.4163], \qquad \mathcal{S}_{2,2} = [0.2717, 1], \qquad \mathcal{S}_{3,2} = [0, 1],$$

respectively. Again, $\mathcal{S}_2 = \mathcal{S}_{1,2} \cap \mathcal{S}_{2,2} \cap \mathcal{S}_{3,2} = [0.2737, 0.4163]$. The conclusion is that A_{22} is a $\mathcal{B}$-matrix and A is a generalized $\mathcal{B}$-matrix. This time we also note that $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2 = [0.3618, 0.4163]$, meaning that we can choose distinct values of α's from the two intervals $\mathcal{S}_1$ and $\mathcal{S}_2$ or a common $\alpha \in \mathcal{S}$. In this last case $\alpha_1 = \alpha_2$ in relations (3.21) which will be strict for all $\alpha \in (0.3618, 0.4163)$ while we will have equalities for $\alpha = 0.3618$ for the cycle $\gamma_{1,1}$ and for $\alpha = 0.4163$ for the cycle $\gamma_{1,2}$.

Based on the theory developed we can give the following statement for the spectrum of the generalized $\mathcal{B}$-matrix A which constitutes an extension of Lemma 2.8. More specifically,

Theorem 3.4. *Under the assumptions and the notation of Definition 3.2 and in view of Remark 3.1 the spectrum of A is contained in the region below*

$$\sigma(A) \subseteq \mathcal{B}(A) := \bigcup_{k=1}^{p} \left\{ z \in \mathbb{C}\colon \prod_{i\in\gamma_{j,k}} |z-a_{ii}| \leqslant \left(\prod_{i\in\gamma_{j,k}} r_i\right)^{\alpha_k} \left(\prod_{i\in\gamma_{j,k}} c_i\right)^{1-\alpha_k} \ (\forall \gamma_{j,k} \in \mathcal{C}_k) \right\}, \quad (3.22)$$

for each p-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_p)$, $\alpha_k \in \mathcal{S}_k$, $k = 1(1)p$.

Remark 3.2. In case of any reducible matrix $A \in \mathbb{C}^{n\times n}$, Theorem 3.4 may be completed adopting Varga's approach [22], where 1×1 blocks in the Frobenius normal form of A are allowed. Then, the spectrum of A is contained in a union of regions of type (3.22), of the singletons whose elements are those of the 1×1 diagonal blocks and of type (3.22), with $\alpha_k = \alpha$, for each $\alpha \in [0, 1]$, for diagonal blocks that are **not** $\mathcal{B}$-matrices, as this is done in Corollary 2.13 of Brualdi [6] (Lemma 2.9).

To make it clear and without loss of generality, we may assume that of the p diagonal blocks A_{kk} of the Frobenius normal form of $A \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, the first p_1 blocks A_{kk}, $k = 1(1)p_1$, are $\mathcal{B}$-matrices, the next p_2 blocks A_{kk}, $k = p_1 + 1(1)p_1 + p_2$ are 1×1 blocks, and the last $p - p_1 - p_2$ blocks A_{kk}, $k = p_1 + p_2 + 1(1)p$, are **not** $\mathcal{B}$-matrices. Then

$$\sigma(A) \subseteq \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3,$$

where

$$\mathcal{E}_1 := \bigcup_{k=1}^{p_1} \left\{ z \in \mathbb{C}\colon \prod_{i\in\gamma_{j,k}} |z-a_{ii}| \leqslant \left(\prod_{i\in\gamma_{j,k}} r_i\right)^{\alpha_k} \left(\prod_{i\in\gamma_{j,k}} c_i\right)^{1-\alpha_k} \text{(for any } \alpha_k \in \mathcal{S}_k,\ \forall \gamma_{j,k} \in \mathcal{C}_k) \right\},$$

$$\mathcal{E}_2 := \bigcup_{k=p_1+1(1)p_1+p_2} \{a_{kk}\},$$

$$\mathcal{E}_3 := \bigcup_{k=p_1+p_2+1}^{p} \left\{ z \in \mathbb{C}\colon \prod_{i\in\gamma_{j,k}} |z-a_{ii}| \leqslant \left(\prod_{i\in\gamma_{j,k}} r_i\right)^{\alpha} \left(\prod_{i\in\gamma_{j,k}} c_i\right)^{1-\alpha} \bigl(\text{for each } \alpha \in [0,1],\ \forall \gamma_{j,k} \in \mathcal{C}_k\bigr) \right\}.$$

4. *M*- and *H*-matrices

In this section we will discuss a little further nonsingular M- and H-matrices, where the term "nonsingular" will be omitted.

Recall that: "*A matrix* $A \in \mathbb{R}^{n\times n}$ *is called an* M*-matrix iff* $a_{ii} > 0$, $a_{ij} \leqslant 0$, $\forall i \neq j \in N$, *and the spectral radius of the associated Jacobi iteration matrix,* $J_A\,(\geqslant 0)$, *is strictly less than* 1. *Namely,* $\rho(J_A) < 1$". Many equivalent characterizations can be found in [1].

In Brualdi [6] a new characterization for M-matrices was given. It goes as follows: "*Let* $A \in \mathbb{R}^{n\times n}$, *with* $a_{ii} > 0$, $a_{ij} \leqslant 0$, $\forall i \neq j \in N$, *and* A *be weakly irreducible. Suppose*

$$\prod_{i\in\gamma_j} a_{ii} > \prod_{i\in\gamma_j} r_i \quad \bigl(\forall \gamma_j \in \mathcal{C}(A)\bigr) \tag{4.1}$$

hold. Then A *is an* M*-matrix*".

For the definition of the "*comparison matrix*" of a given matrix $A \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, we have

Definition 4.1. The comparison matrix of $A \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, is the matrix $\langle A\rangle \in \mathbb{R}^{n\times n}$, with elements $\langle a_{ii}\rangle = |a_{ii}|$, $\langle a_{ij}\rangle = -|a_{ij}|$, $\forall i \neq j \in N$.

The definition for a matrix $A \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, to be an H-matrix, was introduced by Varga [21]. One of its characterizations is: "*A matrix* $A \in \mathbb{C}^{n\times n}$ *is an* H*-matrix iff its comparison matrix is an* M*-matrix*". Obviously, there are many equivalent characterizations for an H-matrix (see [1] and also [5]). A new one can be based on the following statement, where to make things simpler, we assume that $A \in \mathbb{C}^{n\times n}$ is irreducible.

Theorem 4.1. *Let* $A \in \mathbb{C}^{n\times n}$, $n \geqslant 2$, *be irreducible. If* A *is a* $\mathcal{B}$*-matrix, then it is an* H*-matrix.*

Proof. Let A be a $\mathcal{B}$-matrix. Then, in view of Definition 3.1 and Theorem 3.1, A will satisfy relations (3.1), namely

$$\prod_{i\in\gamma_j} |a_{ii}| \geqslant \Bigl(\prod_{i\in\gamma_j} r_i\Bigr)^{\alpha}\Bigl(\prod_{i\in\gamma_j} c_i\Bigr)^{1-\alpha} \quad \bigl(\forall \gamma_j \in \mathcal{C}(A)\bigr) \tag{4.2}$$

for some $\alpha \in [0,1]$ and strict inequality for at least one cycle γ_j. Let $\langle D\rangle = \operatorname{diag}(\langle A\rangle)$, and let $B = \langle D\rangle - \langle A\rangle\,(\geqslant 0)$. Suppose that all the diagonal elements of A are multiplied by a number $\epsilon \in [1,+\infty)$. Take a certain $\epsilon_0 \in (1,+\infty)$ very large so that the new A, $A(\epsilon_0)$, is *SDD* and, therefore, $\langle A(\epsilon_0)\rangle$ is an M-matrix. Then, for any $\epsilon \in (1,\epsilon_0]$ each new A, call it $A(\epsilon)$, will satisfy the corresponding relations to (4.2) and the inequalities will be strict. More specifically,

$$\epsilon^{\beta_j} \prod_{i\in\gamma_j} |a_{ii}| \equiv \prod_{i\in\gamma_j} \bigl(\epsilon|a_{ii}|\bigr) > \Bigl(\prod_{i\in\gamma_j} r_i\Bigr)^{\alpha}\Bigl(\prod_{i\in\gamma_j} c_i\Bigr)^{1-\alpha} \quad \bigl(\forall \gamma_j \in \mathcal{C}(A)\bigr), \tag{4.3}$$

where β_j is the number of nodes on $\gamma_j \in \mathcal{C}(A)$. Therefore, $A(\epsilon)$ will be a $\mathcal{B}$-matrix. Assume that ϵ starts decreasing from ϵ_0 to 1^+. $\langle A(\epsilon)\rangle$ satisfies relations (4.3). Hence, it remains nonsingular. At the limit $\lim_{\epsilon\to 1^+}\langle D(\epsilon)\rangle = \langle D\rangle$, $\lim_{\epsilon\to 1^+}\langle A(\epsilon)\rangle = \langle A\rangle$ and $\lim_{\epsilon\to 1^+} J_{\langle A(\epsilon)\rangle} = J_{\langle A\rangle}$, $\lim_{\epsilon\to 1^+}\rho(J_{\langle A(\epsilon)\rangle}) = \rho(J_{\langle A\rangle})$. However, by Theorem 3.1, $\langle A\rangle$ is nonsingular. From the continuity of all the functions of ϵ involved, $\langle D(\epsilon)\rangle$, $\langle A(\epsilon)\rangle$, $J_{\langle A(\epsilon)\rangle}$, $\rho(J_{\langle A(\epsilon)\rangle})$, we get

$$\begin{aligned} \big\langle D(1)\big\rangle = \langle D\rangle, \qquad & \big\langle A(1)\big\rangle = \langle A\rangle, \\ J_{\langle A(1)\rangle} = J_{\langle A\rangle}, \qquad & \rho(J_{\langle A(1)\rangle}) = \rho(J_{\langle A\rangle}) < 1. \end{aligned} \tag{4.4}$$

These results and the nonsingularity of $\langle A\rangle$ imply that $\langle A\rangle$ is an M-matrix because for **no** value of $\epsilon \in [1, \epsilon_0] \subset [1, +\infty)$, $\langle A(\epsilon)\rangle$ becomes singular. Consequently, A is an H-matrix. □

An immediate consequence of the above theorem is the more general one.

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$, $n \geqslant 4$, be weakly irreducible and a generalized $\mathcal{B}$-matrix which is already in its Frobenius normal form. Then, A is an H-matrix.*

Note: It is reminded that the term *generalized $\mathcal{B}$-matrix* means that

$$\prod_{i\in\gamma_{j,k}} |a_{ii}| \geqslant \prod_{i\in\gamma_{j,k}} r_i^{\alpha_k} c_i^{1-\alpha_k} \quad \big(\forall\gamma_{j,k} \in \mathcal{C}_k,\ \forall\mathcal{C}_k \in \mathcal{C}(A)\big), \tag{4.5}$$

with at least one inequality strict for $\gamma_{j,k} \in \mathcal{C}_k$, $k = 1(1)p$, hold for some p-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_p)$, $\alpha_k \in \mathcal{S}_k \subseteq [0, 1]$, $\mathcal{S}_k \neq \emptyset$, $k = 1(1)p$.

Remark 4.1. It is understood that if $A \in \mathbb{R}^{n\times n}$ has positive diagonal elements and nonpositive off-diagonal ones, the (generalized) $\mathcal{B}$-matrix character of A implies that in both Theorems 4.1 and 4.2 the conclusion is that A is an M-matrix.

Notes: Under the same assumptions as those of Theorems 4.1 and 4.2 for a matrix $A \in \mathbb{C}^{n\times n}$, the theorems of this section can apply to: (i) Reducible matrices A which, in their Frobenius normal form, have nonzero 1×1 diagonal blocks and (ii) matrices $X^{-1}AX$, where $X \in \mathbb{R}^{n\times n}$ is any positive diagonal matrix.

5. Concluding remarks

In the present work we used the theory presented in Brualdi's paper [6] and in the analytically developed theory in the recent book by Varga [22] and were able to determine the value(s) of the parameter(s) α (α_k's) involved mainly in Lemmas 2.8 and 2.9 due to Brualdi, as well as to Theorem 2.10 of [22] due to Brualdi and Varga. This determination was based on previous works by Cvetković et al. [8] and the first of the present authors [12]. Theorem 3.2 together with Table 1, Remark 3.1, Theorem 3.3, Corollary 3.1, Theorem 3.4, Remark 3.2 and Theorems 4.1 and 4.2 are considered as new results. The numerical examples given, together with the depicted figures, make the theory in the paper easier for the reader to follow.

Last but not least we would like to point out that a similar theory can be developed if, instead of the *generalized geometric means*, one considers the *generalized arithmetic means* as this was done in [8] and especially in [12].

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