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Nonoverlapping domain decomposition: A linear algebra viewpoint ☆

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Abstract

In this work we consider the Helmholtz equation in a hyperparallelepiped $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3, \dots$, under Dirichlet boundary conditions and for its solution we apply the averaging technique of the nonoverlapping Domain Decomposition, where Ω is decomposed in two, in general not equal, subdomains. Unlike what many researchers do that is first to determine regions of convergence and optimal values of the relaxation parameters involved at the PDE level, next discretize and then solve the linear system yielded using the values of the parameters determined, we determine regions of convergence and optimal values of the parameters involved *after* the discretization takes place, that is at the linear algebra level, and then use them for the solution of the linear system. In the general case the parameters obtained in this work are *not* the same with the ones which are known and which have been obtained at the PDE level. ©2000 IMACS/Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3, \dots$, be an open convex polygon with boundary $\partial\Omega$. We consider the boundary value problem

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$$Lu = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega \tag{1.1}$$

where L is a linear elliptic operator and f and g known functions.

For the solution of the continuous problem (1.1) a discretized analog (linear algebraic system) is obtained. In practical problems the size of the system is enormous, and so the computing time required for its solution is very large. This time issue and the development of parallel computers led to the idea of splitting up the original problem into a number of smaller ones. Thus methods like the Domain Decomposition (DD) methods have been developed.

The idea of DD with overlapping subdomains at the PDE level goes back to Schwarz [21] (1869). His method is now known as Schwarz Splitting (SS). It was Miller [14], in 1965, who recognized its importance for the numerical solution of PDEs.

In the last 15 years SS has attracted the attention of many researchers who have extended and generalized the basic algorithm (Rodrigue and Simon [20], Rodrigue [19], Oliger, Skamarock and Tang [16]), analyzed the convergence properties (Tang [23,24]), applied it in many important problems and implemented it on computers of parallel architecture (see, e.g., [4,8–10]). At linear algebra level the DD as SS has been studied by few researchers (see, e.g., [10–12,16,19,20,23,24]).

However, the actual performance of the overlapping DD was not quite satisfactory, mainly due to the extra computing because the overlap participates in the solution of neighboring subdomains. So, researchers were led to the consideration of the DD into nonoverlapping subdomains. One can see such efforts in many works (see, e.g., [1,3,5,6,13,17,18,22,26] etc.).

In this work we will study the nonoverlapping DD known as the *averaging technique* at linear algebra level. For this technique let us consider the decomposition of the domain Ω into two subdomains (the technique is extended in an obvious way to consider more subdomains) Ω_1 and Ω_2 with

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \qquad \Omega_1 \cap \Omega_2 = \emptyset, \qquad \partial\Omega_1 \cap \partial\Omega \neq \emptyset, \qquad \Omega_2 \cap \partial\Omega \neq \emptyset. \tag{1.2}$$

Let $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ be the common boundary of the two subdomains. Then the algorithm of the averaging technique at PDE level is as follows:

Algorithm 1.1. *For* $k = 0, 1, 2, \ldots$, *until convergence do*

(i) *Solve* $Lu_1^{(2k+1)} = f$ *in* Ω_1 *with* $u_1^{(2k+1)} = \alpha u_1^{(2k)} + (1-\alpha)u_2^{(2k)}$ *on* Γ.

(ii) *Solve* $Lu_2^{(2k+1)} = f$ *in* Ω_2 *with* $u_2^{(2k+1)} = \alpha u_2^{(2k)} + (1-\alpha)u_1^{(2k)}$ *on* Γ.

(iii) *Solve* $Lu_1^{(2k+2)} = f$ *in* Ω_1 *with* $\frac{\partial u_1^{(2k+2)}}{\partial \nu^1} = \beta \frac{u_1^{(2k+1)}}{\partial \nu^1} + (1-\beta)\frac{\partial u_2^{(2k+1)}}{\partial \nu^1}$ *on* Γ.

(iv) *Solve* $Lu_2^{(2k+2)} = f$ *in* Ω_2 *with* $\frac{\partial u_2^{(2k+2)}}{\partial \nu^2} = \beta \frac{\partial u_2^{(2k+1)}}{\partial \nu^2} + (1-\beta)\frac{\partial u_1^{(2k+1)}}{\partial \nu^2}$ *on* Γ.

End of iteration

In Algorithm 1.1, $\alpha, \beta \in (0, 1)$ are relaxation parameters to be determined so that the iterative procedure converges as fast as possible. Note that the first two problems in the algorithm have Dirichlet boundary conditions on Γ with values of u on the common boundary a convex combination of the up to then available ones. The last two problems have Neumann boundary conditions on Γ with values of the outwardly directed normal derivatives on the common boundary a convex combination of the up to then available. As is seen one solves alternatively a Dirichlet and a mixed boundary value problem in the two subdomains smoothing each time the values of the function and those of the outwardly normal derivative on Γ. This is done until convergence is achieved.

For the determination of the (optimal) parameters involved most of the researchers work at the PDE level. The advantage of working at this level is that the (optimal) parameters that are determined are discretization independent. However, for the solution of the PDE problem by Algorithm 1.1 a discrete equivalent algorithm is applied where the (optimal) values of the parameters of the continuous problem are used for the solution of the discrete one. But these values may *not* be the (optimal) ones that make the discrete iterative algorithm converge as fast as possible. So, it is more natural, although more difficult to analyze and study, to consider the problem of the determination of the (optimal) parameters *after* the discretization takes place.

The first theoretical results for the nonoverlapping DD method at the PDE level for the Helmholtz equation in two-dimensional rectangular domains and two subdomains seem to have been obtained by Rice, Vavalis and Yang [18]. Our objective in this work is to analyze and study in one and two dimensions the same problem at the linear algebra level and then try to extend its study to three and more than three dimensions.

Consider then the Helmholtz equation

$$-\Delta u + qu = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{1.3}$$

where q is a positive constant and Ω a hyperparallelepiped in $\mathbb{R}^d$, $d = 1, 2, 3, \ldots$.

First we will study Eq. (1.3) in the one-dimensional case. From the practical point of view, the study of it seems to be worthless since the solution of the discrete analog of Eq. (1.3) can be obtained with negligible computing cost by using classical methods. The computing cost is a serious issue when one solves problems in two and more than two dimensions. However, as we shall see in the sequel the analysis in the one-dimensional case helps a lot when one moves on to higher dimensional problems. The analysis for the latter ones would be much more difficult if one could not have in mind how the one-dimensional problem is attacked and solved.

In this work we will derive the linear iterative method from the discrete analog to Algorithm 1.1 using finite differences, will study it and will derive regions of convergence and optimal values for the parameters α and β. We will describe the process of extending the method to two dimensions and will derive corresponding regions of convergence as well as optimal values for the relaxation parameters. Finally, an obvious extension will show how to determine regions of convergence and optimal parameters in three and more than three dimensions.

2. One-dimensional case

We consider the two-point boundary value Helmholtz equation

$$-u'' + qu = f \text{ in } \Omega \equiv (0, 1), \qquad u(0) = a \text{ and } u(1) = b, \tag{2.1}$$

where q is a positive constant and a, b given values. We discretize uniformly Ω into $m+n$ subintervals of length $h = 1/(m+n)$. We decompose Ω into two subdomains so that $\Omega_1 \equiv (0, m/(m+n))$ and $\Omega_2 \equiv (m/(m+n), 1)$ as this is shown in Fig. 1.

The discretization of problem (2.1) using second-order finite differences for u'' gives at the node x_i the equation

$$-u_{i-1} + (2 + qh^2)u_i - u_{i+1} = h^2 f_i, \quad i = 1(1)m+n-1, \qquad u_0 = a, \qquad u_{m+n} = b, \tag{2.2}$$

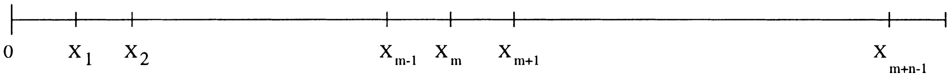

Fig. 1. Discretization of the one-dimensional domain.

where we set $u_i = u(x_i)$ and $f_i = f(x_i)$. The discretization of problem (i) of Algorithm 1.1 yields the $(m-1)\times(m-1)$ linear system

$$\begin{bmatrix} 2+qh^2 & -1 & & \\ -1 & 2+qh^2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2+qh^2 \end{bmatrix} \begin{bmatrix} u_1^{(2k+1)} \\ u_2^{(2k+1)} \\ \vdots \\ u_{m-1}^{(2k+1)} \end{bmatrix} = \begin{bmatrix} h^2 f_1 + a \\ h^2 f_2 \\ \vdots \\ h^2 f_{m-1} + (u_m^{(2k+1)})_1 \end{bmatrix} \tag{2.3}$$

where $(u_m^{(j)})_1$ and $(u_m^{(j)})_2$ are the jth approximate values of u at the boundary node x_m of the left and the right subdomain, respectively. The value $(u_m^{(2k+1)})_1$ according to the condition on Γ of the Dirichlet problem will be given by

$$(u_m^{(2k+1)})_1 = \alpha(u_m^{(2k)})_1 + (1-\alpha)(u_m^{(2k)})_2. \tag{2.4}$$

In an analogous way the discretization of problem (ii) of Algorithm 1.1 yields the following linear system

$$\begin{bmatrix} 2+qh^2 & -1 & & \\ -1 & 2+qh^2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2+qh^2 \end{bmatrix} \begin{bmatrix} u_{m+1}^{(2k+1)} \\ u_{m+2}^{(2k+1)} \\ \vdots \\ u_{m+n-1}^{(2k+1)} \end{bmatrix} = \begin{bmatrix} h^2 f_{m+1} + (u_m^{(2k+1)})_2 \\ h^2 f_{m+2} \\ \vdots \\ h^2 f_{m+n-1} + b \end{bmatrix} \tag{2.5}$$

with

$$(u_m^{(2k+1)})_2 = \alpha(u_m^{(2k)})_2 + (1-\alpha)(u_m^{(2k)})_1. \tag{2.6}$$

Discretizing the problems under conditions (iii) and (iv) of Algorithm 1.1 we will obtain similar to (2.3) and (2.5) linear systems, respectively, with a superscript $(2k+2)$ to u's and with the difference that the boundary conditions admit one more discretization which will give one more equation for each system. In this case $(u_m^{(2k+2)})_1$ and $(u_m^{(2k+2)})_2$ are unknowns and are transferred to the left-hand sides of the corresponding linear systems.

Note that the discretization of u'' was done with a local truncation error of order $O(h^2)$. For consistency the discretization of the first derivatives must be done with a local truncation error of the same order.

Thus we take

$$\frac{\partial u_1^{(j)}}{\partial \nu^1} = \frac{1}{h}\left(\frac{3}{2}(u_m^{(j)})_1 - 2u_{m-1}^{(j)} + \frac{1}{2}u_{m-2}^{(j)}\right) + \mathrm{O}(h^2),$$
$$\frac{\partial u_1^{(j)}}{\partial \nu^2} = \frac{1}{h}\left(-\frac{3}{2}(u_m^{(j)})_2 + 2u_{m+1}^{(j)} - \frac{1}{2}u_{m+2}^{(j)}\right) + \mathrm{O}(h^2) \qquad (2.7)$$

and the boundary condition of problem (iii) of Algorithm 1.1 gives the equation

$$\tfrac{3}{2}(u_m^{(2k+2)})_1 - 2u_{m-1}^{(2k+2)} + \tfrac{1}{2}u_{m-2}^{(2k+2)} = \beta\left[\tfrac{3}{2}(u_m^{(2k+1)})_1 - 2u_{m-1}^{(2k+1)} + \tfrac{1}{2}u_{m-2}^{(2k+1)}\right]$$
$$+(1-\beta)\left[-\tfrac{3}{2}(u_m^{(2k+1)})_2 + 2u_{m+1}^{(2k+1)} - \tfrac{1}{2}u_{m+2}^{(2k+1)}\right].$$

Substituting $(u_m^{(2k+1)})_1$ and $(u_m^{(2k+1)})_2$ from Eqs. (2.4) and (2.6) produces

$$\tfrac{3}{2}(u_m^{(2k+2)})_1 - 2u_{m-1}^{(2k+2)} + \tfrac{1}{2}u_{m-2}^{(2k+2)} = \tfrac{3}{2}(\alpha+\beta-1)(u_m^{(2k)})_1 + \tfrac{3}{2}(\beta-\alpha)(u_m^{(2k)})_2 + \tfrac{1}{2}\beta u_{m-2}^{(2k+1)}$$
$$-2\beta u_{m-1}^{(2k+1)} + 2(1-\beta)u_{m+1}^{(2k+1)} - \tfrac{1}{2}(1-\beta)u_{m+2}^{(2k+1)}. \qquad (2.8)$$

Following the same reasoning, from boundary condition (iv) of Algorithm 1.1, the following equation is obtained

$$\tfrac{3}{2}(u_m^{(2k+2)})_2 - 2u_{m+1}^{(2k+2)} + \tfrac{1}{2}u_{m+2}^{(2k+2)} = \tfrac{3}{2}(\alpha+\beta-1)(u_m^{(2k)})_2 + \tfrac{3}{2}(\beta-\alpha)(u_m^{(2k)})_1 - \tfrac{1}{2}(1-\beta)u_{m-2}^{(2k+1)}$$
$$+2(1-\beta)u_{m-1}^{(2k+1)} - 2\beta u_{m+1}^{(2k+1)} + \tfrac{1}{2}\beta u_{m+2}^{(2k+1)}. \qquad (2.9)$$

So, the discretization of problem (iii) of Algorithm 1.1 gives the linear system

$$\begin{bmatrix} 2+qh^2 & -1 & & & \\ -1 & 2+qh^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+qh^2 & -1 \\ & & \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} u_1^{(2k+2)} \\ u_2^{(2k+2)} \\ \vdots \\ u_{m-1}^{(2k+2)} \\ (u_m^{(2k+2)})_1 \end{bmatrix}$$
$$= \begin{bmatrix} h^2 f_1 + a \\ h^2 f_2 \\ \vdots \\ h^2 f_{m-1} \\ \left\{\tfrac{3}{2}(\alpha+\beta-1)(u_m^{(2k)})_1 + \tfrac{3}{2}(\beta-\alpha)(u_m^{(2k)})_2 + \tfrac{1}{2}\beta u_{m-2}^{(2k+1)} - 2\beta u_{m-1}^{(2k+1)} \right. \\ \quad \left. +2(1-\beta)u_{m+1}^{(2k+1)} - \tfrac{1}{2}(1-\beta)u_{m+2}^{(2k+1)}\right\} \end{bmatrix} \qquad (2.10)$$

while the discretization of problem (iv) of Algorithm 1.1 gives the linear system

$$\begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} & & \\ -1 & 2+qh^2 & -1 & & \\ & -1 & 2+qh^2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2+qh^2 \end{bmatrix} \begin{bmatrix} (u_m^{(2k+2)})_2 \\ u_{m+1}^{(2k+2)} \\ u_{m+2}^{(2k+2)} \\ \vdots \\ u_{m+n-1}^{(2k+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \left\{ \frac{3}{2}(\alpha+\beta-1)(u_m^{(2k)})_2 + \frac{3}{2}(\beta-\alpha)(u_m^{(2k)})_1 - \frac{1}{2}(1-\beta)u_{m-2}^{(2k+1)} \right. \\ \left. \quad +2(1-\beta)u_{m-1}^{(2k+1)} - 2\beta u_{m+1}^{(2k+1)} + \frac{1}{2}\beta u_{m+2}^{(2k+1)} \right\} \\ h^2 f_{m+1} \\ h^2 f_{m+2} \\ \vdots \\ h^2 f_{m+n-1} + b \end{bmatrix}. \tag{2.11}$$

Thus the discretization of Algorithm 1.1 gives the following discrete algorithm.

Algorithm 2.1. *Give arbitrary values to* $u_i^{(0)}$, $i = 1(1)m+n-1,\ i \neq m$, *and* $(u_m^{(0)})_1$, $(u_m^{(0)})_2$.
For $k = 0, 1, 2, \ldots$, *until convergence do*
(i) *Solve system* (2.3) *under condition* (2.4).
(ii) *Solve system* (2.5) *under condition* (2.6).
(iii) *Solve system* (2.10).
(iv) *Solve system* (2.11).
End of iteration

It is quite clear that steps (i) and (ii) as well as steps (iii) and (iv) of Algorithm 2.1 are fully parallelizable. Thus Algorithm 2.1 can be modified as follows:

Algorithm 2.2. *Give arbitrary values to* $u_i^{(0)}$, $i = 1(1)m+n-1,\ i \neq m$, *and* $(u_m^{(0)})_1$, $(u_m^{(0)})_2$.
For $k = 0, 1, 2, \ldots$, *until convergence do*
(i) *Solve in parallel systems* (2.3) *and* (2.5) *under conditions* (2.4) *and* (2.6), *respectively*.
(ii) *Solve in parallel systems* (2.10) *and* (2.1).
End of iteration

So after the discretization we succeeded in transforming the continuous PDE problem into a discrete one of Linear Algebra. It remains then to study the problem at the linear algebra level and determine the possible values of the pairs (α, β) in order to have convergence. To study it we combine the steps of the above iterative process into a classical iterative scheme and the study of the convergence of the latter is made by means of the convergence properties of the corresponding iteration matrix (see, e.g., [2,25,27]).

The four linear systems (2.3), (2.5), (2.10) and (2.11) are combined into the following one:

$$
\left[\begin{array}{cccc|cccc|cccc|cccc}
d & -1 & & & & & & & & & & & & & & \\
-1 & d & -1 & & & & & & & & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & & & & & & & & \\
 & & -1 & d & & & & & & & & & & & & \\
\hline
 & & & & d & -1 & & & & & & & & & & \\
 & & & & -1 & d & -1 & & & & & & & & & \\
 & & & & & \ddots & \ddots & \ddots & & & & & & & & \\
 & & & & & & -1 & d & & & & & & & & \\
\hline
 & & & & & & & & d & -1 & & & & & & \\
 & & & & & & & & -1 & d & -1 & & & & & \\
 & & & & & & & & & \ddots & \ddots & \ddots & & & & \\
 & & & & & & & & & -1 & d & -1 & & & & \\
 & & -\frac{1}{2}\beta & 2\beta & -2(1-\beta) & \frac{1}{2}(1-\beta) & & & & \frac{1}{2} & -2 & \frac{3}{2} & & & & \\
\hline
 & & \frac{1}{2}(1-\beta) & -2(1-\beta) & 2\beta & -\frac{1}{2}\beta & & & & & & & \frac{3}{2} & -2 & \frac{1}{2} & \\
 & & & & & & & & & & & & -1 & d & -1 & \\
 & & & & & & & & & & & & & \ddots & \ddots & \ddots \\
 & & & & & & & & & & & & & -1 & d & -1 \\
 & & & & & & & & & & & & & & -1 & d
\end{array}\right]
\left[\begin{array}{c}
u_1^{(2k+1)} \\ u_2^{(2k+1)} \\ \vdots \\ u_{m-1}^{(2k+1)} \\
\hline
u_{m+1}^{(2k+1)} \\ u_{m+2}^{(2k+1)} \\ \vdots \\ u_{m+n-1}^{(2k+1)} \\
\hline
u_1^{(2k+2)} \\ u_2^{(2k+2)} \\ \vdots \\ u_{m-1}^{(2k+2)} \\ (u_m^{(2k+2)})_1 \\
\hline
(u_m^{(2k+2)})_2 \\ u_{m+1}^{(2k+2)} \\ \vdots \\ u_{m+n-2}^{(2k+2)} \\ u_{m+n-1}^{(2k+2)}
\end{array}\right]
$$

$$
= \left[\begin{array}{cc|cc|cc|cc}
 & & & & & & & \\
 & & & & & \alpha & 1-\alpha & \\
\hline
 & & & & & 1-\alpha & \alpha & \\
 & & & & & & & \\
\hline
 & & & & & & & \\
 & & & & & \frac{3}{2}(\alpha+\beta-1) & \frac{3}{2}(\beta-\alpha) & \\
\hline
 & & & & & \frac{3}{2}(\beta-\alpha) & \frac{3}{2}(\alpha+\beta-1) & \\
 & & & & & & &
\end{array}\right]
\left[\begin{array}{c}
u_1^{(2k-1)} \\ u_2^{(2k-1)} \\ \vdots \\ u_{m-1}^{(2k-1)} \\
\hline
u_{m+1}^{(2k-1)} \\ u_{m+2}^{(2k-1)} \\ \vdots \\ u_{m+n-1}^{(2k-1)} \\
\hline
u_1^{(2k)} \\ u_2^{(2k)} \\ \vdots \\ u_{m-1}^{(2k)} \\ (u_m^{(2k)})_1 \\
\hline
(u_m^{(2k)})_2 \\ u_{m+1}^{(2k)} \\ \vdots \\ u_{m+n-2}^{(2k)} \\ u_{m+n-1}^{(2k)}
\end{array}\right]
+
\left[\begin{array}{c}
h^2 f_1 + a \\ h^2 f_2 \\ \vdots \\ h^2 f_{m-1} \\
\hline
h^2 f_{m+1} \\ h^2 f_{m+2} \\ \vdots \\ h^2 f_{m+n-1} + b \\
\hline
h^2 f_1 + a \\ h^2 f_2 \\ \vdots \\ h^2 f_{m-1} \\ 0 \\
\hline
0 \\ h^2 f_{m+1} \\ \vdots \\ h^2 f_{m+n-2} \\ h^2 f_{m+n-1} + b
\end{array}\right]
\tag{2.12}
$$

where $d=2+qh^2$. Iterative Scheme (2.12) can be written as

$$
Tu^{(k+1)} = Cu^{(k)} + \tilde{f} \tag{2.13}
$$

where T, $C \in \mathbb{R}^{2(m+n-1),2(m+n-1)}$ and $u^{(k+1)}$, $u^{(k)}$, $\tilde{f} \in \mathbb{R}^{2(m+n-1)}$ are the matrices and the vectors in the sequence given in Eq. (2.12). From Eq. (2.12), T and C can be written in the following block form

$$T = \begin{bmatrix} T_{m-1} & 0 & 0 & 0 \\ 0 & T_{n-1} & 0 & 0 \\ B_{31} & B_{32} & \underline{T}_m & 0 \\ B_{41} & B_{42} & 0 & \overline{T}_n \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 & C_{13} & C_{14} \\ 0 & 0 & C_{23} & C_{24} \\ 0 & 0 & C_{33} & C_{34} \\ 0 & 0 & C_{43} & C_{44} \end{bmatrix}. \tag{2.14}$$

As is seen from Eq. (2.12) the matrix C has only nonzero elements in the $(2m+n-2)^{\text{nd}}$ and $(2m+n-1)^{\text{th}}$ columns. In exactly the same columns the iterative matrix

$$S = T^{-1}C \tag{2.15}$$

will have nonzero elements. This means that all the eigenvalues of S will be identically zero except those coming from the 2×2 diagonal block

$$S_2 = \begin{bmatrix} s_{2m+n-2,2m+n-2} & s_{2m+n-2,2m+n-1} \\ s_{2m+n-1,2m+n-2} & s_{2m+n-1,2m+n-1} \end{bmatrix}. \tag{2.16}$$

To determine the elements of S_2, from Eq. (2.14) it is readily seen that T^{-1} is given by

$$T^{-1} = \begin{bmatrix} T_{m-1}^{-1} & 0 & 0 & 0 \\ 0 & T_{n-1}^{-1} & 0 & 0 \\ -\underline{T}_m^{-1} B_{31} T_{m-1}^{-1} & -\underline{T}_m^{-1} B_{32} T_{n-1}^{-1} & \underline{T}_m^{-1} & 0 \\ -\overline{T}_n^{-1} B_{41} T_{m-1}^{-1} & -\overline{T}_n^{-1} B_{42} T_{n-1}^{-1} & 0 & \overline{T}_n^{-1} \end{bmatrix} \tag{2.17}$$

and then

$$\begin{aligned} s_{2m+n-2,2m+n-2} &= (-\underline{T}_m^{-1} B_{31} T_{m-1}^{-1} C_{13} - \underline{T}_m^{-1} B_{32} T_{n-1}^{-1} C_{23} + \underline{T}_m^{-1} C_{33})_{m,m}, \\ s_{2m+n-2,2m+n-1} &= (-\underline{T}_m^{-1} B_{31} T_{m-1}^{-1} C_{14} - \underline{T}_m^{-1} B_{32} T_{n-1}^{-1} C_{24} + \underline{T}_m^{-1} C_{34})_{m,1}, \\ s_{2m+n-1,2m+n-2} &= (-\overline{T}_n^{-1} B_{41} T_{m-1}^{-1} C_{13} - \overline{T}_n^{-1} B_{42} T_{n-1}^{-1} C_{23} + \overline{T}_n^{-1} C_{43})_{1,m}, \\ s_{2m+n-1,2m+n-1} &= (-\overline{T}_n^{-1} B_{41} T_{m-1}^{-1} C_{14} - \overline{T}_n^{-1} B_{42} T_{n-1}^{-1} C_{24} + \overline{T}_n^{-1} C_{44})_{1,1}. \end{aligned} \tag{2.18}$$

Next we compute one by one the elements of the first expression in Eq. (2.18). So

$$\begin{aligned} (-\underline{T}_m^{-1} B_{31} T_{m-1}^{-1} C_{13})_{m,m} &= -\sum_{i=1}^{m-1} (\underline{T}_m^{-1} B_{31})_{m,i} (T_{m-1}^{-1} C_{13})_{i,m} \\ &= -\sum_{i=1}^{m-1} \left(\sum_{j=1}^{m} (\underline{T}_m^{-1})_{m,j} (B_{31})_{j,i} \sum_{l=1}^{m-2} (T_{m-1}^{-1})_{i,l} (C_{13})_{l,m} \right). \end{aligned} \tag{2.19}$$

Since the matrix B_{31} has nonzero elements only in the mth row the index j in the sum will take the value m only. Also, l takes only the value $m-1$ since the matrix C_{13} has only one nonzero element,

$(C_{13})_{m-1,m}=\alpha$. Therefore, Eq. (2.19) will give

$$(-\underline{T}_m^{-1}B_{31}T_{m-1}^{-1}C_{13})_{m,m}=-\sum_{i=1}^{m-1}(\underline{T}_m^{-1})_{m,m}(B_{31})_{m,i}(T_{m-1}^{-1})_{i,m-1}(C_{13})_{m-1,m}$$

and since B_{31} has only $(B_{31})_{m,m-2}$ and $(B_{31})_{m,m-1}$ as its nonzero elements it will be

$$\begin{aligned}(-\underline{T}_m^{-1}B_{31}T_{m-1}^{-1}C_{13})_{m,m}&=(\underline{T}_m^{-1})_{m,m}(B_{31})_{m,m-2}(T_{m-1}^{-1})_{m-2,m-1}(C_{13})_{m-1,m}\\&\quad-(\underline{T}_m^{-1})_{m,m}(B_{31})_{m,m-1}(T_{m-1}^{-1})_{m-1,m-1}(C_{13})_{m-1,m}\\&=\tfrac{1}{2}\alpha\beta(\underline{T}_m^{-1})_{m,m}(T_{m-1}^{-1})_{m-2,m-1}-2\alpha\beta(\underline{T}_m^{-1})_{m,m}(T_{m-1}^{-1})_{m-1,m-1}.\end{aligned}\tag{2.20}$$

To find $(\underline{T}_m^{-1})_{m,m}$ we use finite difference equations. For this we set $2+qh^2=2\cosh\theta$ and if y_i, $i=1(1)m$, are the elements of the mth column of $\underline{T}_m^{-1}$ we will have

$$\begin{aligned}&2\cosh\theta\, y_1-y_2=0\\&-y_1+2\cosh\theta\, y_2-y_3=0\\&\vdots\\&-y_{m-2}+2\cosh\theta\, y_{m-1}-y_m=0\end{aligned}\tag{2.21}$$

Eqs. (2.21) are given by the difference equation

$$-y_{i-1}+2\cosh\theta\, y_i-y_{i+1}=0,\quad i=1(1)m-1\tag{2.22}$$

and the boundary conditions

$$y_0=0\quad\text{and}\quad \tfrac{1}{2}y_{m-2}-2y_{m-1}+\tfrac{3}{2}y_m=1.\tag{2.23}$$

The solution of Eqs. (2.22) and (2.23) is

$$y_i=\frac{2\sinh i\theta}{\sinh(m-2)\theta-4\sinh(m-1)\theta+3\sinh m\theta}\tag{2.24}$$

Therefore,

$$(\underline{T}_m^{-1})_{m,m}=y_m=\frac{2\sinh m\theta}{\sinh(m-2)\theta-4\sinh(m-1)\theta+3\sinh m\theta}.\tag{2.25}$$

Following the same process we find

$$(T_{m-1}^{-1})_{m-2,m-1}=\frac{\sinh(m-2)\theta}{\sinh m\theta}\quad\text{and}\quad(T_{m-1}^{-1})_{m-1,m-1}=\frac{\sinh(m-1)\theta}{\sinh m\theta}.\tag{2.26}$$

Hence, Eq. (2.20) becomes

$$(-\underline{T}_m^{-1}B_{31}T_{m-1}^{-1}C_{13})_{m,m}=\frac{2\alpha\beta((1/2)\sinh(m-2)\theta-2\sinh(m-1)\theta)}{\sinh(m-2)\theta-4\sinh(m-1)\theta+3\sinh m\theta} \tag{2.27}$$

Following the same steps we find all the other terms in the first expression of Eq. (2.18). Thus it is obtained that

$$\begin{aligned} s_{2m+n-2,2m+n-2}=&\frac{2\sinh m\theta}{\sinh(m-2)\theta-4\sinh(m-1)\theta+3\sinh m\theta}\\ &\times\left[\alpha\beta\frac{(1/2)\sinh(m-2)\theta-2\sinh(m-1)\theta}{\sinh m\theta}+\frac{3}{2}(\alpha+\beta-1)\right.\\ &\left.+(1-\alpha)(1-\beta)\frac{2\sinh(n-1)\theta-(1/2)\sinh(n-2)\theta}{\sinh n\theta}\right]. \end{aligned} \tag{2.28}$$

After some manipulation we can obtain that

$$s_{2m+n-2,2m+n-2}=\alpha\beta-(1-\alpha)(1-\beta)p_{m,n}(\theta), \tag{2.29}$$

where

$$p_{m,n}(\theta)=\frac{\sinh m\theta[\sinh(n-2)\theta-4\sinh(n-1)\theta+3\sinh n\theta]}{\sinh n\theta[\sinh(m-2)\theta-4\sinh(m-1)\theta+3\sinh m\theta]}. \tag{2.30}$$

We note that the second expression of Eq. (2.18) differs from the first one only as regards the elements of the matrix C. It is easy to conclude that the corresponding relationship will be produced from Eq. (2.28) if we replace α by $(1-\alpha)$. Thus we obtain

$$s_{2m+n-2,2m+n-1}=(1-\alpha)\beta-\alpha(1-\beta)p_{m,n}(\theta). \tag{2.31}$$

Following the same steps as before we can derive the relationships that the third and fourth expressions of Eq. (2.18) give. However, we observe a symmetry in the problem if we interchange the roles of m and n. So the element $s_{2m+n-1,2m+n-2}$ will be produced from Eq. (2.31) if we interchange m and n, while $s_{2m+n-1,2m+n-2}$ will be produced from Eq. (2.28), in a similar way. More specifically,

$$s_{2m+n-1,2m+n-2}=(1-\alpha)\beta-\alpha(1-\beta)p_{n,m}(\theta), \tag{2.32}$$

$$s_{2m+n-1,2m+n-1}=\alpha\beta-(1-\alpha)(1-\beta)p_{n,m}(\theta). \tag{2.33}$$

The eigenvalues of S_2 are roots of the equation

$$\lambda^2-\mathrm{tr}(S_2)\lambda+\det(S_2)=0 \tag{2.34}$$

with

$$\begin{aligned} \mathrm{tr}(S_2)=&\,2\alpha\beta-(1-\alpha)(1-\beta)(p_{m,n}(\theta)+p_{n,m}(\theta)),\\ \det(S_2)=&\,[\alpha\beta-(1-\alpha)(1-\beta)p_{m,n}(\theta)][\alpha\beta-(1-\alpha)(1-\beta)p_{n,m}(\theta)]\\ &-[(1-\alpha)\beta-\alpha(1-\beta)p_{m,n}(\theta)][(1-\alpha)\beta-\alpha(1-\beta)p_{n,m}(\theta)]\\ =&\,(2\alpha-1)(2\beta-1). \end{aligned} \tag{2.35}$$

We observe that $\det(S_2)$ vanishes for $\alpha=1/2$ or $\beta=1/2$ and $\mathrm{tr}(S_2)$ vanishes for $\beta=(p_{m,n}(\theta)+p_{n,m}(\theta))/(2+p_{m,n}(\theta)p_{n,m}(\theta))$ or $\alpha=(p_{m,n}(\theta)+p_{n,m}(\theta))/(2+p_{m,n}(\theta)p_{n,m}(\theta))$, respectively. So, the (optimal) values of the parameters that make the spectral radius vanish have been found. However, for these values the 2×2 matrix S_2 in its canonical form is associated with a Jordan block of order 2 meaning that although $S_2\neq 0$, $S_2^2=0$. Since the only nonzero elements of S are in the same two columns in which the elements of S_2 are, in view of $S_2^2=0$, for the optimal pair found, it will be $S^3=0$ and so the exact solution of the linear system will be obtained after two iterations. This basic result is given in the following statement.

Theorem 2.1. *For the solution of the two boundary value Helmholtz equation we discretize uniformly the interval of definition and apply the method of decomposing the domain into two nonoverlapping subdomains as this was described previously. Then the optimal pair of the parameters (α,β) are given by*

$$(\alpha,\beta)=\left(\frac{1}{2},\frac{p_{m,n}(\theta)+p_{n,m}(\theta)}{2+p_{m,n}(\theta)p_{n,m}(\theta)}\right)\ \textit{or}\ \left(\frac{p_{m,n}(\theta)+p_{n,m}(\theta)}{2+p_{m,n}(\theta)p_{n,m}(\theta)},\frac{1}{2}\right) \tag{2.36}$$

and the algorithm converges to the exact solution of the linear system in two iterations.

If we choose the two subdomains to be of equal length, namely $\Omega_1=(0,1/2)$, $\Omega_2=(1/2,1)$ and $\Gamma=1/2$ then the application of the previous theorem gives the following corollary.

Corollary 2.1. *For the solution of the problem defined in Theorem* 2.1, *considering equal subdomains* ($m=n$) *the (optimal) pair of the parameters* (α,β) *is* $(1/2, 1/2)$ *and the algorithm converges into one iteration!*

Notes. (a) *The values $\alpha=\beta=1/2$ make all four elements of S_2 vanish. As a result of this $S_2=0$ and the exact values of the unknowns are obtained after only one iteration.* (b) *The results of this statement were also obtained in* [15].

Obviously the theory developed so far holds in the case of Poisson equation as well in which case $q=0$. Then it will be $\theta=0$ and is readily found that $\lim_{\theta\to 0} p_{m,n}(\theta)=(m/n)$. In this case Theorem 2.1 becomes:

Corollary 2.2. *For the solution of the two boundary value Poisson equation we discretize uniformly the interval of definition and apply the method of domain decomposition into two nonoverlapping subdomains as this was described previously. Then, the optimal pair of the parameters* (α,β) *are* $(1/2, (m^2+n^2)/(m+n)^2)$ *or* $((m^2+n^2)/(m+n)^2, 1/2)$, *and the algorithm converges into two iterations.*

Also in the case of the two equal subdomains for the Poisson equation Corollary 2.1 holds the same.

The analysis we have done so far allows us to determine also the values of the parameters α and β for which convergence of the proposed scheme takes place. Thus we have:

Theorem 2.2. *For the solution of the problem that is described in Theorem* 2.1, *the values of α and β (regions of convergence) for which Scheme* (2.12) *converges are*

$$\begin{aligned}K:=\{(\alpha,\beta)\in\mathbb{R}^2 : 0<\alpha<1,\ 0<\beta<1 \text{ and } (6-p_{m,n}(\theta)-p_{n,m}(\theta))\alpha\beta\\ +(p_{m,n}(\theta)+p_{n,m}(\theta)-2)(\alpha+\beta-1)>0\}.\end{aligned} \tag{2.37}$$

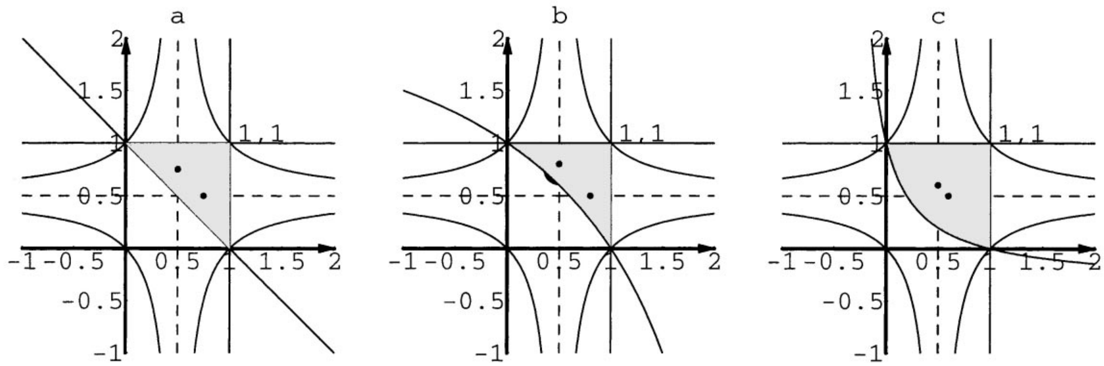

Fig. 2. Regions of convergence.

Proof. To determine the region of convergence we must find the conditions so that the roots of the quadratic (2.34) lie strictly in the interior of the unit disk. An obvious condition is

$$|\det(S_2)| < 1 \Leftrightarrow -1 < (2\alpha - 1)(2\beta - 1) < 1 \tag{2.38}$$

which gives the region between the two hyperbolas $(2\alpha-1)(2\beta-1)=1$ and $(2\alpha-1)(2\beta-1)=-1$, as this is depicted in Fig. 2. This condition covers even the case where the quadratic has complex conjugate roots with modulus less than 1. When, however, the roots are real they must lie in the interval $(-1, 1)$ and the quadratic must take positive values at the points -1 and 1. Therefore, we have also the conditions

$$1 - \mathrm{tr}(S_2) + \det(S_2) > 0 \quad \text{and} \quad 1 + \mathrm{tr}(S_2) + \det(S_2) > 0. \tag{2.39}$$

Here it is noted that the same conditions would have been obtained if we had applied the Schur–Cohn algorithm [7]. If we substitute the values of Eq. (2.35), the first condition gives

$$(1-\alpha)(1-\beta) > 0 \tag{2.40}$$

and the second one

$$(6 - p_{m,n}(\theta) - p_{n,m}(\theta))\alpha\beta + (p_{m,n}(\theta) + p_{n,m}(\theta) - 2)(\alpha + \beta - 1) > 0. \tag{2.41}$$

Conditions (2.38), (2.40) and (2.41) are all satisfied in the region K of Eq. (2.37). □

In Fig. 2a the curves of conditions (2.38) and (2.41) are depicted in the degenerate case when $6 - p_{m,n}(\theta) - p_{n,m}(\theta) = 0$ and Eq. (2.41) becomes $\alpha + \beta - 1 > 0$, in Fig. 2b when $6 - p_{m,n}(\theta) - p_{n,m}(\theta) < 0$ while in Fig. 2c the curves are shown when $6 - p_{m,n}(\theta) - p_{n,m}(\theta) > 0$.

As in the case of the optimal parameters we can also give here analogous statements.

Corollary 2.3. *For the solution of the problem defined in Theorem* 2.1 *and in the case of equal subdomains $m = n$, the region of convergence is given by all the pairs of the parameters (α, β) that lie in the open unit square, that is*

$$K = \{(\alpha, \beta) \in \mathbb{R}^2 : \ 0 < \alpha < 1, \ 0 < \beta < 1\}. \tag{2.42}$$

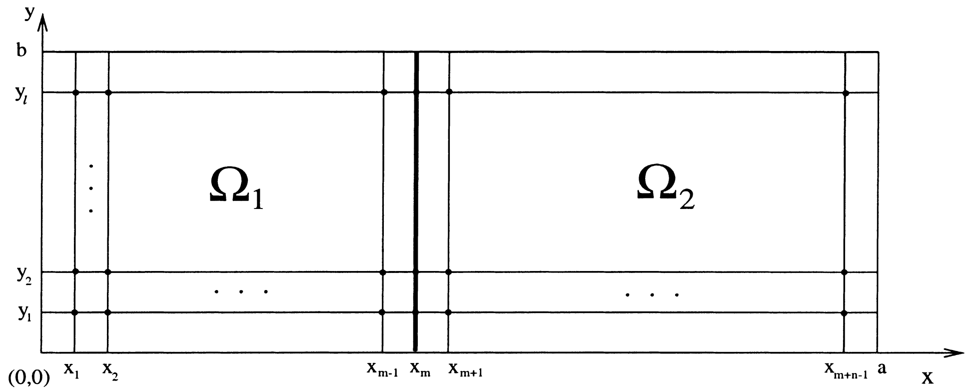

Fig. 3. Discretization of the two-dimensional domain.

Proof. In the present case Eq. (2.41) degenerates to $\alpha\beta > 0$, and the region of convergence is the one given in Eq. (2.42). □

Corollary 2.4. *For the solution of the Poisson equation as this is defined in Corollary* 2.2 *the region of convergence for the parameters* (α, β) *is given by*

$$K = \Big\{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha < 1,\ 0 < \beta < 1 \, and \, \Big(6 - \frac{m}{n} - \frac{n}{m}\Big)\alpha\beta + \Big(\frac{m}{n} + \frac{n}{m} - 2\Big)(\alpha + \beta - 1) > 0\Big\}. \quad (2.43)$$

Corollary 2.5. *For the solution of the problem defined in Corollary* 2.4 *in the case of equal subdomains* ($m = n$)*, the region of convergence for the parameters* (α, β) *is that in Eq.* (2.42) (*open unit square*).

3. Two-dimensional case

We consider the Helmholtz boundary value problem

$$-\Delta u + qu = f \text{ in } \Omega = (0, a) \times (0, b), \quad u = g \text{ on } \partial\Omega \quad (3.1)$$

where q is a positive constant and g is given. We discretize uniformly Ω with $m + n$ subintervals in the x-direction and $l + 1$ in the y-direction assuming that $h = a/(m + n) = b/(l + 1)$. We decompose Ω into two subdomains $\Omega_1 = (0,\ mh) \times (0,\ b)$ and $\Omega_2 = (mh,\ a) \times (0,\ b)$ as this is shown in Fig. 3.

In the discretization we first order the nodes along the y-direction and then along the x-direction. We denote by $U_i^{(k)}$ the ith l-dimensional block element of the iteration vector during the kth iteration, namely $U_i^{(k)} = [u_{(i-1)l+1}^{(k)}, u_{(i-1)l+2}^{(k)}, \ldots, u_{il}^{(k)}]^{\mathrm{T}}$. The discretization of the Dirichlet problem in Ω_1 gives a linear system which in block form corresponds to the one in Eq. (2.3)

$$\begin{bmatrix} D_l & -I_l & & & \\ -I_l & D_l & -I_l & & \\ & \ddots & \ddots & \ddots & \\ & & & & -I_l \\ & & & -I_l & D_l \end{bmatrix} \begin{bmatrix} U_1^{(2k+1)} \\ U_2^{(2k+1)} \\ \vdots \\ U_{m-1}^{(2k+1)} \end{bmatrix} = \begin{bmatrix} F_1 + G_1 \\ F_2 \\ \vdots \\ F_{m-1} + (U_m^{(2k+1)})_1 \end{bmatrix}. \tag{3.2}$$

In the same way the discretization of the Dirichlet problem in Ω_2 gives the linear system

$$\begin{bmatrix} D_l & -I_l & & & \\ -I_l & D_l & -I_l & & \\ & \ddots & \ddots & \ddots & \\ & & & & -I_l \\ & & & -I_l & D_l \end{bmatrix} \begin{bmatrix} U_{m+1}^{(2k+1)} \\ U_{m+2}^{(2k+1)} \\ \vdots \\ U_{m+n-1}^{(2k+1)} \end{bmatrix} = \begin{bmatrix} F_{m+1} + (U_m^{(2k+1)})_2 \\ F_{m+2} \\ \vdots \\ F_{m+n-1} + G_2 \end{bmatrix}, \tag{3.3}$$

where I_l is the $l \times l$ unit matrix and D_l the $l \times l$ matrix

$$D_l = \begin{bmatrix} 4+qh^2 & -1 & & & \\ -1 & 4+qh^2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 4+qh^2 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} g(0, y_1) \\ g(0, y_2) \\ \vdots \\ g(0, y_l) \end{bmatrix},$$

$$G_2 = \begin{bmatrix} g(a, y_1) \\ g(a, y_2) \\ \vdots \\ g(a, y_l) \end{bmatrix} \quad \text{and} \quad F_i = \begin{bmatrix} h^2 f(x_i, y_1) + g(x_i, 0) \\ h^2 f(x_i, y_2) \\ \vdots \\ h^2 f(x_i, y_l) + g(x_i, b) \end{bmatrix}, \quad i = 1(1)m+n+1 \text{ and } i \neq m. \tag{3.4}$$

$(U_m^{(j)})_1$ and $(U_m^{(j)})_2$ are in analogy to the one-dimensional case the jth approximations to u on the common boundary Γ of Ω_1 and Ω_2, respectively. As in the one-dimensional case these values are taken to be the linear combinations

$$(U_m^{(2k+1)})_1 = \alpha(U_m^{(2k)})_1 + (1-\alpha)(U_m^{(2k)})_2, \qquad (U_m^{(2k+1)})_2 = \alpha(U_m^{(2k)})_2 + (1-\alpha)(U_m^{(2k)})_1 \tag{3.5}$$

The discretization of the outwardly normal derivatives on Γ is analogous to that in the one-dimensional case. Taking the corresponding boundary conditions we end up with the one corresponding to (2.10) and (2.11) linear systems

$$\begin{bmatrix} D_l & -I_l & & & \\ -I_l & D_l & -I_l & & \\ & \ddots & \ddots & \ddots & \\ & & -I_l & D_l & -I_l \\ & & \frac{1}{2}I_l & 2I_l & \frac{3}{2}I_l \end{bmatrix} \begin{bmatrix} U_1^{(2k+2)} \\ U_2^{(2k+2)} \\ \vdots \\ U_{m-1}^{(2k+2)} \\ (U_m^{(2k+2)})_1 \end{bmatrix}$$

$$= \begin{bmatrix} F_1 + G_1 \\ F_2 \\ \vdots \\ F_{m-1} \\ \left\{\frac{3}{2}(\alpha+\beta-1)(U_m^{(2k)})_1 + \frac{3}{2}(\beta-\alpha)(U_m^{(2k)})_2 + \frac{1}{2}\beta U_{m-2}^{(2k+1)} - 2\beta U_{m-1}^{(2k+1)} \right. \\ \left. +2(1-\beta)U_{m+1}^{(2k+1)} - \frac{1}{2}(1-\beta)U_{m+2}^{(2k+1)}\right\} \end{bmatrix} \quad (3.6)$$

and

$$\begin{bmatrix} \frac{3}{2}I_l & -2I_l & \frac{1}{2}I_l & & & \\ -I_l & D_l & -I_l & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -I_l & D_l & -I_l \\ & & & & -I_l & D_l \end{bmatrix} \begin{bmatrix} U_m^{((2k+2))_2} \\ U_{m+1}^{(2k+2)} \\ \vdots \\ U_{m+n-2}^{(2k+2)} \\ U_{m+n-1}^{(2k+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \left\{\frac{3}{2}(\alpha+\beta-1)(U_m^{(2k)})_2 + \frac{3}{2}(\beta-\alpha)(U_m^{(2k)})_1 - \frac{1}{2}(1-\beta)U_{m-2}^{(2k+1)} \right. \\ \left. +2(1-\beta)U_{m-1}^{(2k+1)} - 2\beta U_{m+1}^{(2k+1)} + \frac{1}{2}\beta U_{m+2}^{(2k+1)}\right\} \\ F_{m+1} \\ \vdots \\ F_{m+n-2} \\ F_{m+n-1} + G_2 \end{bmatrix}. \quad (3.7)$$

Based on the above the two-dimensional problem is solved with the following parallel algorithm.

Algorithm 3.1. *Give arbitrary values to* $U_i^{(-1)}$, $U_i^{(0)}$, $i=1(1)m+n-1$, $i \neq \mathrm{m}$, *and* $(U_m^{(0)})_1$, $(U_m^{(0)})_2$.
For $k=0,1,2,\ldots,$ *until convergence do*
(i) *Solve in parallel linear systems* (3.2) *and* (3.3) *subject to conditions* (3.4).
(ii) *Solve in parallel linear systems* (3.6) *and* (3.7) *subject to conditions analogous to the ones in Eq.* (2.9) *for the two-dimensional case.*
End of iteration

For the study of the convergence of the problem we combine the four linear systems into one and study the corresponding iterative scheme. After this combination takes place we obtain

$$
\left[\begin{array}{cccc|cccc|cccc|cccc}
D_l & -I_l & & & & & & & & & & & & & & \\
-I_l & D_l & -I_l & & & & & & & & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & & & & & & & & \\
 & & -I_l & D_l & & & & & & & & & & & & \\
\hline
 & & & & D_l & -I_l & & & & & & & & & & \\
 & & & & -I_l & D_l & -I_l & & & & & & & & & \\
 & & & & & \ddots & \ddots & \ddots & & & & & & & & \\
 & & & & & & -I_l & D_l & & & & & & & & \\
\hline
 & & & & & & & & D_l & -I_l & & & & & & \\
 & & & & & & & & -I_l & D_l & -I_l & & & & & \\
 & & & & & & & & & \ddots & \ddots & \ddots & & & & \\
 & & & & & & & & & -I_l & D_l & -I_l & & & & \\
 & & -\frac{1}{2}\beta I_l & 2\beta I_l & -2(1-\beta)I_l & \frac{1}{2}(1-\beta)I_l & & & & \frac{1}{2}I_l & -2I_l & \frac{3}{2}I_l & & & & \\
\hline
 & & \frac{1}{2}(1-\beta)I_l & -2(1-\beta)I_l & 2\beta I_l & -\frac{1}{2}\beta I_l & & & & & & & \frac{3}{2}I_l & -2I_l & \frac{1}{2}I_l & \\
 & & & & & & & & & & & & -I_l & D_l & -I_l & \\
 & & & & & & & & & & & & & \ddots & \ddots & \ddots \\
 & & & & & & & & & & & & & -I_l & D_l & -I_l \\
 & & & & & & & & & & & & & & -I_l & D_l
\end{array}\right]
\left[\begin{array}{c}
U_1^{(2k+1)} \\ U_2^{(2k+1)} \\ \vdots \\ U_{m-1}^{(2k+1)} \\
\hline
U_{m+1}^{(2k+1)} \\ U_{m+2}^{(2k+1)} \\ \vdots \\ U_{m+n-1}^{(2k+1)} \\
\hline
U_1^{(2k+2)} \\ U_2^{(2k+2)} \\ \vdots \\ U_{m-1}^{(2k+2)} \\ (U_m^{(2k+2)})_1 \\
\hline
(U_m^{(2k+2)})_2 \\ U_{m+1}^{(2k+2)} \\ \vdots \\ U_{m+n-2}^{(2k+2)} \\ (U_{m+n-1}^{(2k+2)})
\end{array}\right]
$$

$$
= \left[\begin{array}{c|c|c|c}
 & & & \\
 & & \alpha I_l & (1-\alpha)I_l \\
\hline
 & & (1-\alpha)I_l & \alpha I_l \\
 & & & \\
\hline
 & & & \\
 & & \frac{3}{2}(\alpha+\beta-1)I_l & \frac{3}{2}(\beta-\alpha)I_l \\
\hline
 & & \frac{3}{2}(\beta-\alpha)I_l & \frac{3}{2}(\alpha+\beta-1)I_l \\
 & & &
\end{array}\right]
\left[\begin{array}{c}
U_1^{(2k-1)} \\ U_2^{(2k-1)} \\ \vdots \\ U_{m-1}^{(2k-1)} \\
\hline
U_{m+1}^{(2k-1)} \\ U_{m+2}^{(2k-1)} \\ \vdots \\ U_{m+n-1}^{(2k-1)} \\
\hline
U_1^{(2k)} \\ U_2^{(2k)} \\ \vdots \\ U_{m-1}^{(2k)} \\ (U_m^{(2k)})_1 \\
\hline
(U_m^{(2k)})_2 \\ U_{m+1}^{(2k)} \\ \vdots \\ U_{m+n-2}^{(2k)} \\ U_{m+n-1}^{(2k)}
\end{array}\right]
+
\left[\begin{array}{c}
F_1 + G_1 \\ F_2 \\ \vdots \\ F_{m-1} \\
\hline
F_{m+1} \\ F_{m+2} \\ \vdots \\ F_{m+n-1} + G_2 \\
\hline
F_1 + G_l \\ F_2 \\ \vdots \\ F_{m-1} \\ 0 \\
\hline
0 \\ F_{m+1} \\ \vdots \\ F_{m+n-2} \\ F_{m+n-1} + G_2
\end{array}\right]
\qquad (3.8)
$$

To study the convergence of iterative Scheme (3.8) we denote again by T and C the matrices that are present in it and by S the matrix $T^{-1}C$.

Let $X\in\mathbb{R}^{l,l}$ be the matrix with columns containing the normalized eigenvectors of D_l. Since D_l is real symmetric, X will be orthonormal. The Jordan canonical form of D_l will be

$$J_l = X^{\mathrm{T}} D_l X \tag{3.9}$$

where $J_l=\mathrm{diag}(\lambda_1, \lambda_2,\dots,\lambda_1)$ with λ_i, $i=1(1)n$, the eigenvalues of D_l which are

$$\lambda_i = 2 + qh^2 + 4\sin^2\frac{i\pi}{2(l+1)}, \quad i = 1(1)l. \tag{3.10}$$

We consider the block diagonal matrix $\tilde{X}=\mathrm{diag}(X, X, \dots, X)$ with $2(m+n)$ diagonal blocks. It is obvious that $\tilde{X}$ will be orthonormal too. Hence, $\tilde{X}^{-1}=\tilde{X}^{\mathrm{T}}$. Considering the similarity transformations of T, C and S with similarity matrix $\tilde{X}$ and recalling that the block elements of T are D_l and I_l while those of C are I_l, then after the transformation the matrices I_l remain unchanged while D_l become J_l. Thus we obtain

$$\tilde{T} = \tilde{X}^{\mathrm{T}} T \tilde{X}, \qquad C = \tilde{X}^{\mathrm{T}} C \tilde{X}$$

and

$$\tilde{S} = \tilde{X}^{\mathrm{T}} T^{-1} C \tilde{X} = \tilde{T}^{-1} C \tag{3.11}$$

where $\tilde{T}$ has the form of T with J_l in the place of D_l. We now consider the permutation transformation of matrix $\tilde{T}$ which is produced from the permutation

$$\begin{aligned} P = \{&1, l+1, 2l+1, \dots, 2(m+n-1)l \\ &+1, 2, 2l+2, \dots, 2(m+n-1)l+2, \\ &\dots, l, 2l, \dots, 2(m+n)l\}. \end{aligned} \tag{3.12}$$

Using it we take the first elements form all $J_l^{'}s$ and place them into the first $2(m+n)\times 2(m+n)$ diagonal block, the second elements in the second diagonal block etc. In other words it is the permutation that recorders the nodes first along the x-direction and then along the y-direction. Thus the matrix $\tilde{T}$ becomes similar to

$$\tilde{T}' = \mathrm{diag}\,(T_1, T_{2,\dots,}T_l) \tag{3.13}$$

where T_i, $i=1(1)l$, is given by

$$T_i = \left[\begin{array}{cccccccccccccccccc}
\lambda_i & -1 &&&&&&&&&&&&&&&& \\
-1 & \lambda_i & -1 &&&&&&&&&&&&&&& \\
& \ddots & \ddots & \ddots &&&&&&&&&&&&&& \\
&& -1 & \lambda_l &&&&&&&&&&&&&& \\
&&&& \lambda_i & -1 &&&&&&&&&&&& \\
&&&& -1 & \lambda_i & -1 &&&&&&&&&&& \\
&&&&& \ddots & \ddots & \ddots &&&&&&&&&& \\
&&&&&& -1 & \lambda_l &&&&&&&&&& \\
&&&&&&&& \lambda_i & -1 &&&&&&&& \\
&&&&&&&& -1 & \lambda_i & -1 &&&&&&& \\
&&&&&&&&& \ddots & \ddots & \ddots &&&&&& \\
&&&&&&&&&& -1 & \lambda_i & -1 &&&&& \\
& -\frac{1}{2}\beta & 2\beta && -2(1-\beta) & \frac{1}{2}(1-\beta) &&&&& \frac{1}{2} & -2 & \frac{3}{2} &&&&& \\
& \frac{1}{2}(1-\beta) & -2(1-\beta) && 2\beta & -\frac{1}{2}\beta &&&&&&&& \frac{3}{2} & -2 & \frac{1}{2} && \\
&&&&&&&&&&&&& -1 & \lambda_i & -1 && \\
&&&&&&&&&&&&&& \ddots & \ddots & \ddots & \\
&&&&&&&&&&&&&&& -1 & \lambda_i & -1 \\
&&&&&&&&&&&&&&&& -1 & \lambda_i
\end{array}\right] \quad (3.14)$$

We note that T_i is the same matrix as T of the one-dimensional case (2.12) with the only difference being that to the diagonal elements d of the latter $4\sin^2(i\pi/2(l+1))$ is added. The same permutation matrix acting on C transforms it into a diagonal matrix with diagonal blocks exactly the matrix C of the one-dimensional case (2.12). So $\tilde{S}$ is transformed into a block diagonal matrix with diagonal blocks S_i which are of the same form as the matrix S of the one-dimensional case. If we put

$$2\cosh\theta_i = 2 + qh^2 + 4\sin^2\frac{i\pi}{2(l+1)}, i = 1(1)l, \quad (3.15)$$

then one can develop the theory of the one-dimensional case for each diagonal block of $\tilde{S}$. Therefore, the nonidentically zero eigenvalues of $\tilde{S}$ will be $2l$ and will be given in pairs from the quadratics

$$\lambda^2 - [2\alpha\beta + (\alpha+\beta-1-\alpha\beta)B_i]\lambda + (2\alpha-1)(2\beta-1) = 0,\ i = 1(1)l, \quad (3.16)$$

as this is implied from Eqs. (2.34) and (2.35), where we have put

$$B_i = p_{m,n}(\theta_i) + p_{n,m}(\theta_i). \quad (3.17)$$

Now we can state and prove statements analogous to the ones in the one-dimensional case regarding the regions of convergence and the optimal values of the parameters α and β. Starting with the regions of convergence the corresponding statement to Theorem 2.2 will be:

Theorem 3.1. *For the solution of the two-dimensional Helmholtz equation under Dirichlet boundary conditions, we uniformly discretize and apply the method of DD as this was described previously. The region of convergence for the parameters* α *and* β *will be*

$$K = \{(\alpha,\beta) \in \mathbb{R}^2 : 0 < \alpha < 1,\ 0 < \beta < 1,\ 4\alpha\beta - (B_{\mathrm{M}} - 2)(1-\alpha)(1-\beta) > 0\}, \quad (3.18)$$

where

$$B_{\mathrm{M}} = \max_i B_i.$$

Proof. From Theorem 2.2 we have that

$$K = \bigcap_{i=1}^{n} K_i = \bigcap_{i=1}^{n} \{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha < 1,\ 0 < \beta < 1,\ 4\alpha\beta - (B_i - 2)(1-\alpha)(1-\beta) > 0\}. \tag{3.19}$$

We observe that K_i differ from each other only in their last condition. We also note that the left-hand side of the last inequality is a decreasing function of B_i. Therefore, the inequality will hold for all $B_i^{'}s$ as long as there holds for $B_{\mathrm{M}} = \max_i B_i$. □

Theorem 3.2. *For the problem of Theorem* 3.1, *the optimal pair of parameters* (α, β) *are*

$$\left(\frac{1}{2}\left[1 + \left(\frac{\sqrt{B_{\mathrm{M}}+2} - \sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}-2} - \sqrt{B_{\mathrm{m}}-2}}\right)^2\right],\ \frac{1}{2}\left[1 + \left(\frac{\sqrt{B_{\mathrm{M}}+2} - \sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}-2} + \sqrt{B_{\mathrm{m}}-2}}\right)^2\right]\right) \quad or$$

$$\left(\frac{1}{2}\left[1 + \left(\frac{\sqrt{B_{\mathrm{M}}+2} - \sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}-2} + \sqrt{B_{\mathrm{m}}-2}}\right)^2\right],\ \frac{1}{2}\left[1 + \left(\frac{\sqrt{B_{\mathrm{M}}+2} - \sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}-2} - \sqrt{B_{\mathrm{m}}-2}}\right)^2\right]\right) \tag{3.20}$$

while the corresponding optimal spectral radius of the iteration matrix S is

$$\rho(S) = \frac{\sqrt{B_{\mathrm{M}}+2} - \sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}+2} + \sqrt{B_{\mathrm{m}}+2}} \tag{3.21}$$

and where

$$B_{\mathrm{m}} = \min_i B_i, \qquad B_{\mathrm{M}} = \max_i B_i. \tag{3.22}$$

Proof. We have to solve a two-parameter optimization problem. Such problems occur very often in the iterative methods and are very difficult to solve. Usually ‘good’ values of the parameters are found instead of optimal ones. In our case, however, we will find optimal parameters. Since the problem is symmetric with respect to its parameters α and β we may assume that $\alpha \geq \beta$. For $i = 1(1)l$, we denote by

$$\lambda_i^+ = \tfrac{1}{2}\left[2\alpha\beta - (1-\alpha)(1-\beta)B_i + \sqrt{[2\alpha\beta - (1-\alpha)(1-\beta)B_i]^2 - 4(2\alpha-1)(2\beta-1)}\right] \tag{3.23}$$

$$\lambda_i^- = \tfrac{1}{2}\left[2\alpha\beta - (1-\alpha)(1-\beta)B_i - \sqrt{[2\alpha\beta - (1-\alpha)(1-\beta)B_i]^2 - 4(2\alpha-1)(2\beta-1)}\right] \tag{3.24}$$

the roots of the quadratics (3.16) which are also the eigenvalues of S. It is easy to note that if λ_i is that out of Eqs. (3.23) and (3.24) that corresponds to the maximum modulus then it is a decreasing function with respect to B_i as long as λ_i^+ and λ_i^- are real. The modulus remains constant for those B_i for which λ_i^+, λ_i^- are complex conjugate numbers. Also, we note that

$$s_i = 2\alpha\beta - (1-\alpha)(1-\beta)B_i \tag{3.25}$$

is a decreasing function of B_i. So, if the spectral radius corresponds to a negative eigenvalue, that will be λ_{M}^- while if it corresponds to a positive one, it will be λ_{m}^+. We investigate a little further these two quantities. First, we assume that the optimal value of the spectral radius, $\rho(S)$, of S corresponds to λ_{M}^{-}. Then there will be α and β in their region of definition such that

$$\frac{\partial\lambda_{\mathrm{M}}^{-}}{\partial\alpha}=\frac{1}{2}\frac{\left[(2\beta+(1-\beta)B_{\mathrm{M}})\sqrt{s_{\mathrm{M}}^{2}-4(2\alpha-1)(2\beta-1)}-s_{\mathrm{M}}\right]+4(2\beta-1)}{\sqrt{s_{\mathrm{M}}^{2}-4(2\alpha-1)(2\beta-1)}}=0,$$

$$\frac{\partial\lambda_{\mathrm{M}}^{-}}{\partial\beta}=\frac{1}{2}\frac{[(2\alpha+(1-\alpha)B_{\mathrm{M}})\sqrt{s_{\mathrm{M}}^{2}-4(2\alpha-1)(2\beta-1)}-s_{\mathrm{M}}]+4(2\alpha-1)}{\sqrt{s_{\mathrm{M}}^{2}-4(2\alpha-1)(2\beta-1)}}=0$$

or

$$(2\beta+(1-\beta)B_{\mathrm{M}})(-\lambda_{\mathrm{M}}^{-})+2(2\beta-1)=0,\qquad (2\alpha+(1-\alpha)B_{\mathrm{M}})(-\lambda_{\mathrm{M}}^{-})+2(2\alpha-1)=0. \tag{3.26}$$

Eliminating λ_{M}^{-} from Eq. (3.26) we obtain

$$(B_{\mathrm{M}}+2)(\alpha-\beta)=0\Leftrightarrow\alpha=\beta. \tag{3.27}$$

However, for $\alpha=\beta$, Eq. (3.26) give

$$(2\alpha+(1-\alpha)B_{\mathrm{M}})(-\lambda_{\mathrm{M}}^{-})+2(2\alpha-1)=0. \tag{3.28}$$

Since the first term of the first member of Eq. (3.28) is positive, α must be strictly less than 1/2. Consequently, $-\lambda_{\mathrm{M}}^{-}>\sqrt{|\lambda_{\mathrm{M}}^{-}\lambda_{\mathrm{M}}^{+}|}=1-2\alpha$, so

$$\begin{aligned}&(2\alpha+(1-\alpha)B_{\mathrm{M}})(-\lambda_{\mathrm{M}}^{-})+2(2\alpha-1)>(2\alpha+(1-\alpha)B_{\mathrm{M}})(1-2\alpha)+2(2\alpha-1)\\&\quad=(1-2\alpha)(B_{\mathrm{M}}-2)(1-\alpha)>0.\end{aligned} \tag{3.29}$$

This means that the optimal value does not correspond to a local minimum of λ_{M}^{-}. If we assume that it corresponds to a local maximum of λ_{m}^{+}, then following a similar reasoning we end up with the same conclusion. Therefore, if the optimal value corresponds to real eigenvalues it will correspond to a point where the maximum goes from λ_{M}^{-} to λ_{m}^{-}, or vice versa, implying that $\lambda_{\mathrm{m}}^{+}=-\lambda_{\mathrm{M}}^{-}$. So, we have to minimize λ_{m}^{+} under the assumption that $\lambda_{\mathrm{m}}^{+}=-\lambda_{\mathrm{M}}^{+}$.

Using Lagrange multipliers we will have

$$\frac{\partial}{\partial\alpha}(\lambda_{\mathrm{m}}^{+}+\mu(\lambda_{\mathrm{m}}^{+}+\lambda_{\mathrm{M}}^{-}))=0,\qquad \frac{\partial}{\partial\beta}(\lambda_{\mathrm{m}}^{+}+\mu(\lambda_{\mathrm{m}}^{+}+\lambda_{\mathrm{M}}^{-}))=0 \tag{3.30}$$

First we examine the case $\partial(\lambda_{\mathrm{m}}^{+}+\lambda_{\mathrm{M}}^{-})/\partial\alpha=\partial(\lambda_{\mathrm{m}}^{+}+\lambda_{\mathrm{M}}^{-})/\partial\beta=0$. Then Eq. (3.30) become $\partial\lambda_{\mathrm{m}}^{+}/\partial\alpha=\partial\lambda_{\mathrm{m}}^{+}/\partial\beta=0$, which can be proved, in a similar way as in the previous case of λ_{M}^{-}, that it can not happen. Therefore, there exists no local extreme value that comes from a critical point meaning that the minimum value will be assumed on the boundary of the region of definition. The region of definition is a subset of the region of convergence, which was given in Theorem 3.1, so that $\lambda_{\mathrm{m}}^{+}=-\lambda_{\mathrm{M}}^{-}\in\mathbb{R}$. If α and β are on the boundary of the region of convergence they will give a spectral radius equal to one and that will be a maximum. Therefore, they must be taken at the other end of the boundary where they go from real to complex conjugate roots, namely when they become double roots. Hence, the optimal pair will be obtained for

$$\Delta_{\rm m} = \Delta_{\rm M} = 0 \quad \text{and} \quad s_{\rm m} = -s_{\rm M} \tag{3.31}$$

where $\Delta_{\rm m}$ and $\Delta_{\rm M}$ denote the corresponding discriminants of the quadratics. Let that $(\partial/\partial\alpha)(\lambda_{\rm m}^{+}+\lambda_{\rm M}^{-}) \neq 0$ or $(\partial/\partial\beta)(\lambda_{\rm m}^{+}+\lambda_{\rm M}^{-}) \neq 0$. Eliminating μ in Eq. (3.30) we find

$$\frac{\partial\lambda_{\rm m}^{+}}{\partial\alpha}\frac{\partial\lambda_{\rm M}^{-}}{\partial\beta} - \frac{\partial\lambda_{\rm m}^{+}}{\partial\beta}\frac{\partial\lambda_{\rm M}^{-}}{\partial\alpha} = 0. \tag{3.32}$$

Making the necessary substitutions and performing all the calculations in Eq. (3.32) we find out that this is verified for $\alpha = \beta$. For the determination of the optimal parameters we put α in the place of β, in the equation $\lambda_{\rm M}^{+} + \lambda_{\rm m}^{+} = 0$. From this we obtain that it suffices to have

$$s_{\rm m} + s_{\rm M} = 0 \Leftrightarrow 4\alpha^2 - (1-\alpha)^2(B_{\rm m} + B_{\rm M}) = 0. \tag{3.33}$$

The solution of Eq. (3.33) gives that the optimal parameter is

$$\alpha = \frac{\sqrt{B_{\rm m} + B_{\rm M}}}{\sqrt{B_{\rm m} + B_{\rm M}} + 2}. \tag{3.34}$$

If we put the value just obtained into the discriminant $\Delta_{\rm m}$, we find out that $\Delta_{\rm m} < 0$ which contradicts our assumption that the roots are real. This means that there is no local minimum in the region of definition. So, the optimal values will be given at the endpoint where they were given in the previous case that is Eq. (3.31) holds.

The case that remains to be examined is when the optimal parameters are obtained when all the eigenvalues are complex. Then the eigenvalues will lie on a circle centered at the origin 0 whose radius will be $\sqrt{(2\alpha-1)(2\beta-1)}$, and so we have to minimize this value or equivalently the function

$$f(\alpha,\beta) = (2\alpha-1)(2\beta-1)(>0), \tag{3.35}$$

where because of the symmetry of the problem, either $1>\alpha\geq\beta>1/2$ or $1/2>\alpha\geq\beta>0$. As we saw in Eq. (3.25), the real parts of $(1/2)s_i$ of λ_i can be ordered from the smallest $(1/2)s_{\rm M}$ to the largest $(1/2)s_{\rm m}$. Assume that we have found the optimal values (α^*, β^*) so that the corresponding s_i^*'s satisfy the inequalities

$$-2\sqrt{f(\alpha^*,\beta^*)} < s_{\rm M}^* < \cdots < s_i^* < \cdots < s_{\rm m}^* < 2\sqrt{f(\alpha^*,\beta^*)}. \tag{3.36}$$

We consider an $\epsilon>0$ small enough, that can increase α^* by ϵ and decrease β^* by ϵ, so that either $1>\alpha^*+\epsilon>\beta^*-\epsilon>1/2$ or $1/2>\alpha^*+\epsilon>\beta^*-\epsilon>0$, whichever applies. We have that

$$\begin{aligned} f(\alpha^*+\epsilon,\beta^*-\epsilon) &= (2\alpha^*-1+2\epsilon)(2\beta^*-1-2\epsilon) \\ &= (2\alpha^*-1)(2\beta^*-1) - 4\epsilon(\alpha^*-\beta^*) - 4\epsilon^2 < f(\alpha^*,\beta^*). \end{aligned} \tag{3.37}$$

On the other hand

$$\begin{aligned} s_i(\alpha^*+\epsilon,\beta^*-\epsilon) &= 2(\alpha^*+\epsilon)(\beta^*-\epsilon) - (1-\alpha^*-\epsilon)(1-\beta^*+\epsilon)B_i \\ &= s_i^* + (B_i-2)\epsilon(\alpha^*-\beta^*+\epsilon) > s_i^* \end{aligned} \tag{3.38}$$

Due to the strict inequalities at the two ends of Eq. (3.36) we can find an ϵ small enough such that $-2\sqrt{f(\alpha^*+\epsilon,\beta^*-\epsilon)} < s_{\rm M}(\alpha^*+\epsilon,\beta^*-\epsilon)$ and $s_{\rm m}(\alpha^*+\epsilon,\beta^*-\epsilon) < 2\sqrt{f(\alpha^*+\epsilon,\beta^*-\epsilon)}$. In this way we improve the spectral radius which contradicts the assumption that the pair (α^*, β^*) is the optimal one. Therefore, for the optimal pair there will hold either $s_{\mathrm{M}}^* = -2\sqrt{f(\alpha^*, \beta^*)}$ or $s_{\mathrm{m}}^* = 2\sqrt{f(\alpha^*, \beta^*)}$. In the following we examine only the case $s_{\mathrm{m}} = 2\sqrt{f(\alpha, \beta)}$. The other case $s_{\mathrm{M}} = -2\sqrt{f(\alpha, \beta)}$ can be examined similarly and can give the same results. So, we consider

$$\Delta_{\mathrm{m}} = 0 \tag{3.39}$$

and try to minimize $f \equiv f(\alpha, \beta)$ under the assumption (3.39) using Lagrange multipliers. We will have

$$\frac{\partial}{\partial\alpha}(f + \mu\Delta_{\mathrm{m}}) = 0, \qquad \frac{\partial}{\partial\beta}(f + \mu\Delta_{\mathrm{m}}) = 0 \tag{3.40}$$

or equivalently

$$\begin{aligned}&2(2\beta - 1) + 2\mu\{[2\beta + (1-\beta)B_{\mathrm{m}}][2\alpha\beta - (1-\alpha)(1-\beta)B_{\mathrm{m}}] - 4(2\beta - 1)\} = 0,\\ &2(2\alpha - 1) + 2\mu\{[2\alpha + (1-\alpha)B_{\mathrm{m}}][2\alpha\beta - (1-\alpha)(1-\beta)B_{\mathrm{m}}] - 4(2\alpha - 1)\} = 0.\end{aligned} \tag{3.41}$$

If we assume that $\partial\Delta_{\mathrm{m}}/\partial\alpha = \partial\Delta_{\mathrm{m}}/\partial\beta = 0$ then Eq. (3.41) will give $\alpha = \beta = 1/2$, when, however, $\partial\Delta_{\mathrm{m}}/\partial\alpha \neq 0$ and $\partial\Delta_{\mathrm{m}}/\partial\beta \neq 0$. Therefore, we examine only the case $\partial\Delta_{\mathrm{m}}/\partial\alpha \neq 0$ or $\partial\Delta_{\mathrm{m}}/\partial\beta \neq 0$. Eliminating μ, Eq. (3.41) give after some manipulation that

$$[2\alpha\beta - (1-\alpha)(1-\beta)B_{\mathrm{m}}](B_{\mathrm{m}} + 2)(\beta - \alpha) = 0 \tag{3.42}$$

which is equivalent to $\alpha = \beta$ since the first factor is $s_{\mathrm{m}} > 0$. To determine α we put $\beta = \alpha$ in Eq. (3.39) which then becomes

$$\begin{aligned}&[2\alpha^2 - (1-\alpha)^2 B_{\mathrm{m}}]^2 - 4(2\alpha - 1)^2 = 0 \Leftrightarrow\\ &(B_{\mathrm{m}} - 2)(\alpha - 1)^2[(B_{\mathrm{m}} - 2)\alpha^2 - 2(B_{\mathrm{m}} + 2)\alpha + (B_{\mathrm{m}} + 2)] = 0.\end{aligned} \tag{3.43}$$

The double root $\alpha = 1$ is discarded since it does not belong to (0, 1). We are left with the only root of Eq. (3.43) which is in the region of definition

$$\alpha = \frac{B_{\mathrm{m}} + 2 - 2\sqrt{B_{\mathrm{m}} + 2}}{B_{\mathrm{m}} - 2} = \frac{\sqrt{B_{\mathrm{m}} + 2}(\sqrt{B_{\mathrm{m}} + 2} - 2)}{(\sqrt{B_{\mathrm{m}} + 2} - 2)(\sqrt{B_{\mathrm{m}} + 2} + 2)} = \frac{\sqrt{B_{\mathrm{m}} + 2}}{\sqrt{B_{\mathrm{m}} + 2} + 2} \tag{3.44}$$

If we put this value into s_{m}, we readily see that $s_{\mathrm{m}} < 0$, which means that the optimal value given by Eq. (3.44) is outside the region of definition since then all the eigenvalues will be real. Therefore, there is no optimal value in the domain of definition. So, the optimal value will be on the boundary and the critical point will be that where λ_{M} becomes complex from real and therefore, λ_{M} is a double root. One case is when $\alpha = \beta = 1$ when the spectral radius is 1 and is therefore discarded. This end of the boundary gives the maximum value. The other end which is when

$$\Delta_{\mathrm{m}} = \Delta_{\mathrm{M}} = 0, \qquad s_{\mathrm{m}} = -s_{\mathrm{M}} \tag{3.45}$$

will certainly give the minimum. We note that Eqs. (3.31) and (3.45) are exactly the same. This means that the optimal subset of the real eigenvalues λ_{M} and λ_{m} and the optimal subset of the complex eigenvalues coincide and they both lie on the boundary of the subsets, that is where they all become double ones. To determine α and β so that Eq. (3.31) hold we use for Δ_{m} and Δ_{M} the expressions

$$\begin{aligned}\Delta_{\mathrm{m}} &= [2(\alpha+\beta-1)-(1-\alpha)(1-\beta)(B_{\mathrm{m}}-2)^2]-4(2\alpha-1)(2\beta-1)\\ &= (1-\alpha)^2(1-\beta)^2(B_{\mathrm{m}}-2)^2-4(\alpha+\beta-1)(1-\alpha)(1-\beta)(B_{\mathrm{m}}-2)+4(\alpha-\beta)^2,\\ \Delta_{\mathrm{M}} &= (1-\alpha)^2(1-\beta)^2(B_M-2)^2-4(\alpha+\beta-1)(1-\alpha)(1-\beta)(B_M-2)+4(\alpha-\beta)^2. \end{aligned} \tag{3.46}$$

In view of Eq. (3.45), expressions (3.46) imply that the quantities $B_{\mathrm{m}}-2$ and $B_{\mathrm{M}}-2$ will be roots of the same quadratic. Let s be their sum and p their product. These will be given from the expressions

$$s=\frac{4(\alpha+\beta-1)}{(1-\alpha)(1-\beta)}, \qquad p=\frac{4(\alpha-\beta)^2}{(1-\alpha)^2(1-\beta)^2}\left(\sqrt{p}=\frac{2(\alpha-\beta)}{(1-\alpha)(1-\beta)}\right). \tag{3.47}$$

From Eq. (3.47) we obtain

$$\frac{s}{\sqrt{p}}=\frac{2(\alpha+\beta-1)}{\alpha-\beta} \quad\text{and}\quad \beta=\frac{s-2\sqrt{p}}{s+2\sqrt{p}}\alpha+\frac{2\sqrt{p}}{s+2\sqrt{p}}. \tag{3.48}$$

Substituting Eq. (3.48) into the first of Eq. (3.47) we have

$$s(1-\alpha)\left(1-\frac{s-2\sqrt{p}}{s+2\sqrt{p}}\alpha-\frac{2\sqrt{p}}{s+2\sqrt{p}}\right)=4\left(\alpha+\frac{s-2\sqrt{p}}{s+2\sqrt{p}}\alpha+\frac{2\sqrt{p}}{s+2\sqrt{p}}-1\right) \tag{3.49}$$

or equivalently

$$(s-2\sqrt{p})\alpha^2-2(s-\sqrt{p}+4)\alpha+s+4=0 \tag{3.50}$$

when

$$\alpha=\frac{s-\sqrt{p}+4-\sqrt{p+4s+16}}{s-2\sqrt{p}}. \tag{3.51}$$

Substituting the expressions for s and p into Eq. (3.51) we have

$$\begin{aligned}\alpha &= \frac{B_{\mathrm{m}}+B_{\mathrm{M}}-\sqrt{(B_{\mathrm{m}}-2)(B_{\mathrm{M}}-2)}-\sqrt{(B_{\mathrm{m}}+2)(B_{\mathrm{M}}+2)}}{(\sqrt{B_{\mathrm{M}}-2}-\sqrt{B_{\mathrm{m}}-2})^2}\\ &= \frac{(1/2)[(B_{\mathrm{m}}-2)+(B_{\mathrm{M}}-2)-2\sqrt{(B_{\mathrm{m}}-2)(B_{\mathrm{M}}-2)}]}{(\sqrt{B_{\mathrm{M}}-2}-\sqrt{B_{\mathrm{m}}-2})^2}\\ &\quad+\frac{(1/2)[(B_{\mathrm{m}}+2)+(B_{\mathrm{M}}+2)-2\sqrt{(B_{\mathrm{m}}+2)(B_{\mathrm{M}}+2)}]}{(\sqrt{B_{\mathrm{M}}-2}-\sqrt{B_{\mathrm{m}}-2})^2}\\ &= \frac{1}{2}\left[1+\left(\frac{\sqrt{B_{\mathrm{M}}+2}-\sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}-2}-\sqrt{B_{\mathrm{m}}-2}}\right)^2\right]. \end{aligned} \tag{3.52}$$

Substituting Eq. (3.52) into the last of Eq. (3.48) gives β

$$\begin{aligned}\beta &= \left(\frac{\sqrt{B_{\mathrm{M}}-2}-\sqrt{B_{\mathrm{m}}-2}}{\sqrt{B_{\mathrm{M}}-2}+\sqrt{B_{\mathrm{m}}-2}}\right)^2\frac{1}{2}\left[1+\left(\frac{\sqrt{B_{\mathrm{M}}+2}-\sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}-2}-\sqrt{B_{\mathrm{m}}-2}}\right)^2\right]+\frac{2\sqrt{(B_{\mathrm{M}}-2)(B_{\mathrm{m}}-2)}}{(\sqrt{B_{\mathrm{M}}-2}+\sqrt{B_{\mathrm{m}}-2})^2}\\ &= \frac{1}{2}\left[1+\left(\frac{\sqrt{B_{\mathrm{M}}+2}-\sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}-2}+\sqrt{B_{\mathrm{m}}-2}}\right)^2\right]. \end{aligned} \tag{3.53}$$

Since the problem is symmetric α and β can be interchanged. So, the spectral radius is given by the following expression

$$\rho(S)=\sqrt{(2\alpha-1)(2\beta-1)}=\frac{\sqrt{B_{\mathrm{M}}+2}-\sqrt{B_{\mathrm{m}}+2}}{\sqrt{B_{\mathrm{M}}+2}+\sqrt{B_{\mathrm{m}}+2}}. \tag{3.54}$$

This concludes the proof of the theorem. □

Remarks.

(i) *The determination of the optimal values completes the study of the two-dimensional problem in the case of two subdomains. Here we note that in the case* $m=n$ *the theorem just proved gives* $(\alpha, \beta)=(1/2, 1/2)$ *and* $\rho(S)=0$ *and the Corollary* 2.1 *holds in this case too. We also point out that the case of the Poisson equation does not give any different results as this happened in the one-dimensional case. Poisson equation is treated as a Helmholtz equation with* $q=0$. *In other words statements corresponding to the ones in the one-dimensional case (e.g., Corollary* 2.2*) do not hold any more.*

(ii) *A study of* B_i *as a function of* $\theta_i\in(0,\infty)$ *reveals that it is a strictly decreasing one in the interval* $(0, \operatorname{arccosh} 2)$ *(from* $(m/n)+(n/m)$ *to* 2*), where at* $\operatorname{arccosh} 2$ *it assumes its minimum value* 2. *Then for a small interval of* θ_i *it is strictly increasing, assumes a maximum value (very close to* 2*) and, finally, strictly decreases and tends asymptotically to* 2. *Since for small values of* h, qh^2 *is small then from Eq.* (3.15) $\operatorname{arccosh} 2$ *is contained in the smallest interval that covers the spectrum of all* θ_i*'s so it will not be unrealistic if we consider as* B_{m} *and* B_{M} *the values of* B_l *and* B_1 *or* 2 *and* B_1, *respectively. In the latter case we have* $\alpha=\beta$. *In all the cases* α *and* β *are very close to each other and close to* 1/2.

4. Three- and higher-dimensional cases

We consider the Helmholtz equation under Dirichlet boundary conditions

$$-\Delta u+qu=f \text{ in } \Omega=(0,a)\times(0,b)\times(0,c),\quad u=g \text{ on } \partial\Omega \tag{4.1}$$

where Δ is the three-dimensional Laplace operator, q a positive constant and g a known function. We discretize uniformly Ω subdividing it into $m+n$ subintervals in the x-direction, l_1+1 in the y-direction and l_2+1 in the z-direction. Assuming that $h=(a/(m+n))=(b/(l_1+1))=(c/(l_2+1))$ we decompose Ω into two subdomains in the x-direction taking m subintervals in Ω_1 and n in Ω_2. Thus, $\Omega_1=(0, mh)\times(0, b)\times(0,c)$ and $\Omega_2=(mh,a)\times(0,b)\times(0,c)$. In the discretization we order the nodes first in the y-direction then in the z-direction and finally in the x-direction.

We can very easily realize how one can go on from the two- to the three-dimensional case applying exactly the same analysis as before. Relationship (3.2) still holds with the only difference that in the place of D_l we have a matrix that is yielded from the presence of the extra two dimensions. Namely, if H_l is the matrix in the case of the one-dimensional Laplace equation, Eq. (3.4) gives

$$D_l=(2+qh^2)I_l+H_l \tag{4.2}$$

In our case we will have the analog of the three-dimensional case, namely

$$D_{l_1l_2}=(2+qh^2)I_{l_2l_2}+I_{l_2}\otimes H_{l_1}+H_{l_2}\otimes I_{l_1} \tag{4.3}$$

In an analogous way all the other entities can be created with no further problem. In this way the analog to iterative Scheme (3.10) is created where in the place of D_l and I_l we now have $D_{l_1l_2}$ and $I_{l_1l_2}$, respectively. It is known that the eigenvalues of $D_{l_1l_2}$ are given by

$$\lambda_{ij} = 2 + qh^2 + 4\sin^2\frac{i\pi}{2(l_1+1)} + 4\sin^2\frac{j\pi}{2(l_2+1)}, \quad i = 1(1)l_1,\ j = 1(1)l_2. \tag{4.4}$$

Applying to $D_{l_1l_2}$ a similarity permutation transformation similar to the one in Eq. (3.9) and subsequently the corresponding permutation similarity transformation to the iterative matrix we end up with a block diagonal matrix of the form (3.13) where the number of blocks is l_1l_2 while each block is of the form (3.14) with λ_{ij} in the place of λ_i. From this point on the theory is developed in exactly the same way. In the place of Eq. (3.15) we now have

$$2\cosh\theta_{ij} = 2 + qh^2 + 4\sin^2\frac{i\pi}{2(l_1+1)} + 4\sin^2\frac{j\pi}{2(l_2+1)}, \quad i = 1(1)l_1,\ \ j = 1(1)l_2. \tag{4.5}$$

The only issue is that of changing the notation which becomes a little more complicated without any other essential change. The smallest eigenvalue will be λ_{11} and the largest one $\lambda_{l_1l_2}$. Finally, if we put

$$B_{\mathrm{m}} = \min_{i,j}\{p_{m,n}(\theta_{ij}) + p_{n,m}(\theta_{ij})\}, \qquad B_{\mathrm{M}} = \max_{i,j}\{p_{m,n}(\theta_{ij}) + p_{n,m}(\theta_{ij})\} \tag{4.6}$$

the conclusions of Theorems 3.1 and 3.2 will hold exactly the same. This concludes, in brief, the three-dimensional case.

In higher dimensions we can go on in exactly the same way. The main difference will always be the formula that will give the eigenvalues. In d dimensions the eigenvalues will be given by

$$\lambda_{i_1i_2\ldots i_{d-1}} = 2 + qh^2 + 4\sum_{j=1}^{d-1}\sin^2\frac{i_j\pi}{2(l_j+1)}, \quad i_j = 1(1)l_j,\ \ j = 1(1)d-1. \tag{4.7}$$

Finally, B_{m} and B_{M} will be given from formulas analogous to Eqs. (4.6) and (4.7) and Theorems 3.1 and 3.2 will hold the same.

With the above extension and generalization the study of the method of decomposing the domain into two nonoverlapping subdomains and using the averaging technique as this was described has been completed.

5. Numerical examples

In order to confirm the validity of the theory developed and also to compare our results against the best available ones obtained at the PDE level we consider the two two-dimensional characteristic examples worked out in the article by Rice, Vavalis and Yang [18]. In [18] the problems considered are the following two PDEs:

Example 1. The Poisson equation (3.1) in the open unit square $\Omega \equiv (0, 1) \times (0, 1)$ with $q = 0$ and the functions f and g being such that the PDE equation has the solution $u(x, y) = \sin((\pi/2)x)y(1-y)$.

Example 2. The Helmholtz equation (3.1) in the open unit square $\Omega \equiv (0, 1) \times (0, 1)$ with $q = 0.5$ and the functions f and g being such that the PDE equation has the solution $u(x, y) = 3\,\mathrm{e}^{x+y}\,x(1-x)y(1-y)$.

Table 1
Example 1: grid size 1/30 × 1/30

x_m	α_{opt}	β_{opt}	$\rho_{\text{opt}}(S)$	Absolute errors
0.5	0.5	0.5	0	$4.07E-6$
0.6	0.500423	0.500423	$8.46741E-4$	$1.27E-2$ $2.28E-5$ $3.56E-6$
0.4	0.500423	0.500423	$8.46741E-4$	$9.23E-3$ $1.56E-5$ $4.43E-6$

We considered the same uniform discretization as in [18] with mesh sizes $h=1/30$ and 1/60. Since the local truncation error is of order $O(h^2)$ this will be of order O(0.00111...) and O(0.000277...), respectively, for the two mesh sizes considered. In other words the first truncation error is of the order of accuracy of two decimal places while the second one is of three decimal places. We used FORTRAN programs with single precision arithmetic and the stopping criterion $||u^{(k+1)}-u^{(k)}||_\infty \leq \epsilon$, with $u^{(k+1)}$, $u^{(k)}$ the two successive iterates of Eq. (2.13), where $\epsilon = 0.5\times 10^{-3}$ and 0.5×10^{-4}, for the two sizes considered. As is seen we required an accuracy of one more decimal place than what the order of the local truncation error suggests. In all the experiments that were worked out the initial guess $u^{(0)}$ was taken to be zero. To find the solution of each of the four linear subsystems in each iteration the method of LU decomposition for banded matrices was used.

In the illustrative tables the following items are exhibited: the position of the interface (x_m), the optimal values of the two parameters involved (α_{opt}, β_{opt}) as well as the corresponding optimal spectral radius

Table 2
Example 1: grid size 1/60 × 1/60

x_m	α_{opt}	β_{opt}	$\rho_{\text{opt}}(S)$	Absolute errors
0.5	0.5	0.5	0	$1.88E-5$
0.6	0.500422	0.500422	$8.44897E-4$	$1.27E-2$ $3.74E-5$ $3.56E-5$
0.4	0.500423	0.500423	$8.46741E-4$	$9.21E-3$ $2.70E-5$ $1.52E-5$
0.65	0.501116	0.501116	$2.23203E-3$	$2.23E-2$ $1.16E-4$ $1.84E-5$ $1.90E-5$
0.35	0.501116	0.501116	$2.23203E-3$	$1.37E-2$ $7.05E-5$ $1.51E-5$ $1.52E-5$

Table 3
Example 2: grid size 1/30 × 1/30

x_m	α_{opt}	β_{opt}	$\rho_{\mathrm{opt}}(S)$	Absolute errors
0.5	0.5	0.5	0	$2.60E-4$
0.6	0.500382	0.500382	$7.64738E-4$	$3.25E-2$ $2.79E-4$ $2.43E-4$
0.4	0.500382	0.500382	$7.64738E-4$	$2.66E-2$ $2.96E-4$ $2.77E-4$

($\rho_{\mathrm{opt}}(S)$), obtained by our theory (see Eqs. (3.20) and (3.21)), and finally the absolute errors $||u^{(k)}-u||_\infty$, where u is the theoretical solution of the given PDE, for each $k=1,2,\ldots$, until the convergence criterion is satisfied.

Looking very carefully at Tables 1–4 one can make the following observations.

(i) When $x_m=0.5$, that is the interface decomposes Ω into two equal subdomains Ω_1 and Ω_2, convergence is achieved in exactly one iteration as the theory developed predicts.

(ii) When $x_m\neq 0.5$, in all the cases examined $\alpha_{\mathrm{opt}}=\beta_{\mathrm{opt}}$ at least for the accuracy sought. This is due to the fact that the quantity B_{m} of Eqs. (2.29), (3.15), (3.17) and (3.22) is very close to 2 as a result of which the two parameters are almost equal as this was explained in Remark (ii) that followed Theorem 3.2.

(iii) In most of the cases where $x_m\neq 0.5$ convergence is achieved after three iterations. Looking at the errors observed one could say that the solution had already been obtained after the second iteration.

Table 4
Example 2: grid size 1/60 × 1/60

x_m	α_{opt}	β_{opt}	$\rho_{\mathrm{opt}}(S)$	Absolute errors
0.5	0.5	0.5	0	$1.05E-4$
0.6	0.500381	0.500381	$7.62953E-4$	$3.26E-2$ $1.44E-4$ $9.92E-5$
0.4	0.500381	0.500381	$7.62953E-4$	$2.67E-2$ $1.32E-4$ $1.07E-4$
0.65	0.501015	0.501015	$2.03071E-3$	$5.46E-2$ $3.10E-4$ $1.04E-4$ $1.04E-4$
0.35	0.501015	0.501015	$2.03071E-3$	$4.04E-2$ $2.41E-4$ $1.08E-4$ $1.08E-4$

So, convergence takes place at an earlier stage. The extra iteration(s) needed to satisfy the stopping criterion can be explained by the presence of the round-off errors and the single precision arithmetic used.

(iv) In the pairs of cases where the interface is at x_m, and $1-x_m$, respectively, the differences in the values of the corresponding optimal parameters obtained are negligible. According to our theory, this is due to the symmetry of the two problems (Theorem 3.2).

(v) As one can check the results in our experiments compared to the corresponding ones in [18], which are the best ones among those in a number of comparable methods, enjoy a better accuracy in all the cases tested.

6. Concluding remarks

As the reader may have realized a most important problem will be that of decomposing the domain into more than two nonoverlapping subdomains. In this general case a preliminary analysis shows that some matrices called Centrosymmetric play a vital role and the study of their properties has a tremendous interest from the Linear Algebra point of view. We have been studying these matrices in order to be able to find regions of convergence and/or optimal parameters in the case of more than two subdomains.

Another possible direction of further research is to study the nonoverlapping DD method as this was described earlier for more general Elliptic PDEs.

We have been investigating all these issues and the very first results so far are very encouraging.

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References

[1] V.I. Agoshkov, Poincare–Stekov’s operators and domain decomposition methods in finite dimensional spaces, in: R. Glowinski, G.H. Golub, G. Meurant, J. Periaux (Eds.), Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia, PA, 1988, pp. 73–112.

[2] A. Berman, R.J. Plemmons, Nonnegative matrices in the mathematical sciences, in: Classics in Applied Mathematics, SIAM, Philadelphia, PA, 1994.

[3] P.E. Bjorstad, O. Widlund, To overlap or not overlap: a note on a domain decomposition method for elliptic problems, SIAM J. Sci. Stat. Comput. 10 (1989) 1053–1061.

[4] C.C. Christara, A. Hadjidimos, E.N. Houstis, J.R. Rice, E.A. Vavalis, Geometry decomposition based methods for solving elliptic PDEs, in: Computational Methods in Flow Analysis, Vol. 2, University of Okayama, Okayama, 1988, pp. 175–182.

[5] D. Funaro, A. Quarteroni, P. Zanolli, An iterative procedure with interface relaxation for domain decomposition methods, SIAM J. Numer. Anal. 6 (1988) 1213–1236.

[6] W. Heinrichs, Domain decomposition for fourth-order problems, SIAM J. Numer. Anal. 30 (1993) 435–453.

[7] P. Henrici, Applied and Computational Complex Analysis, Wiley, New York, 1974.

[8] E.N. Houstis, J.R. Rice, E.A. Vavalis, A Schwarz splitting variant of cubic spline collocation for elliptic PDEs, in: 3rd Conference on Hupercube Concurrent Computers and Applications, Vol. II, 1988, pp. 1746–1754.
[9] D. Keyes, W. Gropp, A comparison of domain decomposition techniques for elliptic partial differential equations and their parallel implementation, SIAM J. Sci. Stat. Comput. 8 (1987) 166–202.
[10] S.-B. Kim, Parallel Iterative Methods. Ph.D. Thesis, Department of Mathematics, Purdue University, West Lafayette, IN, 1994.
[11] S.-B. Kim, A. Hadjidimos, E.N. Houstis, J.R. Rice, Multi-parameterized Schwarz alternating methods for elliptic boundary value problems, Math. Comput. Simulation 42 (1996) 47–76.
[12] Y.-L. Lai, A. Hadjidimos, E.N. Houstis, A generalized Schwarz splitting method based on Hermite collocation for elliptic boundary value problems, Appl. Numer. Math. 21 (1996) 265–290.
[13] P.L. Lions, On the Schwarz alternating method: a variant for nonoverlapping subdomains, in: R. Glowinski, G.H. Golub, G. Meurant, J. Periaux (Eds.), Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia, PA, 1988, pp. 202–223.
[14] K. Miller, Numerical analogs of the Schwarz alternating procedure, Numer. Math. 7 (1965) 91–103.
[15] I. Moultisas, Domain decomposition methods for elliptic PDEs: a linear algebra view-point, Dissertation Thesis, Department of Mathematics, University of Crete, Heraklion, Greece, 1997.
[16] J. Oliger, W. Skamarock, W.P. Tang, Schwarz alternating methods and its SOR accelerations, Technical Report, Computer Science Department, Stanford University, Stanford, CA, 1986.
[17] J.R. Rice, P. Tsompanopoulou, E.A. Vavalis, Review and performance of interface relaxation methods for elliptic PDEs. Technical Report CSD-TR-96-113, Department of Computer Sciences, Purdue University, West Lafayette, IN, 1996.
[18] J.R. Rice, E.A. Vavalis, D. Yang, Convergence analysis of a nonoverlapping domain decomposition method for elliptic PDEs, J. Comput. Appl. Math. 87 (1997) 11–19.
[19] G. Rodrigue, Some ideas for decomposing the domain of elliptic partial differential equation in the Schwarz process, Commun. Appl. Numer. Methods 2 (1986) 245–249.
[20] G. Rodrigue, I. Simon, A generalization of the numerical Schwarz algorithm, in: R. Glowiski, J.L. Lions (Eds.), Computing Methods in Applied Sciences and Engineering VI, North-Holland, Amsterdam, 1984, pp. 273–283.
[21] H.A. Schwarz, Uber einige Abbildungsaufgaben, J. Reine Angew. Math. 70 (1869) 105–120.
[22] P. Le Tallec, Y. De Roeck, M. Vidrascu, Domain decomposition methods for large linearly elliptic three-dimensional problems, J. Comput. Appl. Math. 34 (1991) 93–117.
[23] W.P. Tang, Schwarz splitting and template operators. Ph.D. Thesis, Department of Computer Science, Stanford University, Stanford, CA, 1987.
[24] W.P. Tang, Generalized Schwarz splittings, SIAM J. Sci. Stat. Comput. 13 (1992) 573–595.
[25] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.
[26] D. Yang, A parallel nonoverlapping Schwarz domain decomposition method for elliptic interface problems, Technical Report IMA # 1508, University of Minnesota, Minneapolis, MN, 1997.
[27] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.