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ON ITERATIVE SOLUTION FOR LINEAR COMPLEMENTARITY PROBLEM WITH AN H_+-MATRIX*

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Abstract. The numerous applications of the *linear complementarity problem* (*LCP*) in, e.g., the solution of linear and convex quadratic programming, free boundary value problems of fluid mechanics, and moving boundary value problems of economics make its efficient numerical solution a very imperative and interesting area of research. For the solution of the LCP, many iterative methods have been proposed, especially, when the matrix of the problem is a real positive definite or an H_+-matrix. In this work we assume that the real matrix of the LCP is an *H_+-matrix* and solve it by using a new method, the *scaled extrapolated block modulus algorithm,* as well as an improved version of the very recently introduced *modulus-based matrix splitting modified AOR iteration method*. As is shown by numerical examples, the two new methods are very effective and competitive with each other.

Key words. LCP, P-matrices, real positive definite matrices, M-matrices, H_+-matrices, strictly diagonally dominant matrices, iterative schemes, scaled extrapolation, (block) modulus algorithm, modulus-based matrix splitting iteration methods, modified AOR method

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1. Introduction and preliminaries.

The linear complementarity problem (LCP) is encountered in numerous applications. (The reader is referred to the reference books [11], [10], and [26].) To define the LCP only the concept of the matrix inequality relation is needed: Two matrices $A, B \in \mathbb{R}^{m,n}$ satisfy $A \geq B\,(A > B)$ or, equivalently, $A - B \geq O\,(A - B > O)$, where O denotes the zero matrix, iff $a_{ij} \geq b_{ij}\,(a_{ij} > b_{ij})\ \forall\, i = 1(1)m,\ j = 1(1)n$. Based on the introduced relation we have the definition for the LCP (see, e.g., [11], [10], or [26]) as follows:

Linear complementarity problem (LCP): Determine $x \in \mathbb{R}^n$, satisfying the conditions

$$r := Ax + q \geq 0, \quad x \geq 0, \quad r^T x = 0 \quad \textit{with} \quad A \in \mathbb{R}^{n,n}, \quad q \in \mathbb{R}^n. \tag{1.1}$$

A sufficient and necessary condition for the LCP to possess a unique solution, $\forall\ q \in \mathbb{R}^n$, is that $A \in \mathbb{R}^{n,n}$ is a *P-matrix* (a matrix with all its principal minors positive). Subclasses of P-matrices are the real positive definite matrices (see [22]) and the *real H-matrices* with positive diagonals (H_+-matrices; see Bai [5]). In the present work we focus on *H_+-matrices*.

Some terminology used throughout the paper follows.

Given $A \in \mathbb{C}^{m,n}$, $|A|$ denotes the matrix whose elements are the moduli of the corresponding elements of A. The *comparison matrix* of $A \in \mathbb{C}^{n,n}$, denoted by $\langle A \rangle = (\langle a_{ij} \rangle)$, is the matrix with $\langle a_{ij} \rangle = |a_{ij}|\ \forall\, i = j = 1(1)n$, and $\langle a_{ij} \rangle = -|a_{ij}|\ \forall\, i \neq j$,

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$i, j = 1(1)n$. $A \in \mathbb{R}^{n,n}$ is a *nonsingular M-matrix* iff $a_{ii} > 0$, $i = 1(1)n$, $a_{ij} \leq 0$, $i \neq j$, $i, j = 1(1)n$, and the Jacobi iteration matrix associated with A converges, i.e., $\rho(I - (\mathrm{diag}(A))^{-1}A) < 1$, with $\rho(\cdot)$ denoting spectral radius. $A \in \mathbb{C}^{n,n}$ is a *nonsingular H-matrix* iff its comparison matrix $\langle A \rangle$ is a nonsingular M-matrix. $A \in \mathbb{R}^{n,n}$ is a *nonsingular H_+-matrix* iff it is a real nonsingular H-matrix with a positive diagonal, i.e., $a_{ii} > 0 \; \forall \, i = 1(1)n$ (see Bai [5]). $A \in \mathbb{C}^{n,n}$ is a *strictly diagonally dominant (SDD) (by rows) matrix* iff $|a_{ii}| > \sum_{j=1,\, j\neq i}^{n} |a_{ij}| \; \forall \, i = 1(1)n$. An SDD matrix is a (nonsingular) H-matrix. Also, it can be proved that, given a nonsingular H-matrix A, there exists a diagonal matrix D, with positive diagonal ($d_{ii} > 0 \; \forall \, i = 1(1)n$), such that AD is SDD [31]. Given $A \in \mathbb{C}^{n,n}$, $A = M - N$ is called a *splitting* of A iff $\det(M) \neq 0$. For various kinds of splittings, such as *regular splittings*, *M-splittings*, *H-splittings*, and *H-compatible splittings*, see Lemma 3.2, Definition 3.3, and Theorem 3.4 in Frommer and Szyld [16] and the references therein.

Iterative methods for the solution of LCP (1.1) were considered first by Cryer [12] and then by Mangasarian [24], Ahn [1], Pang [28], and others. Recently, a growing interest has been shown in them. In some of the works, the idea of multisplitting, introduced by O'Leary and White [27], was used by, e.g., Bai [4], [5], Bai and coauthors [7], [8], [9], and others, who applied it to the parallel solution of the LCP. In other works, various kinds of iterative methods were used (see, e.g., [23], [32], [13], [14], [6]). In this work we are concerned with two new methods: (i) the *scaled extrapolated block modulus algorithm (SEBMA)*, an extension of a recent work [18] by two of the present authors, and (ii) an improvement of the most recent *modulus-based matrix splitting iteration method*, where the splitting is that of the *modified AOR method*, introduced by Bai [6].

In section 2, we analyze and develop SEBMA. In section 3, we study further the *modulus-based matrix splitting modified AOR method*, determine regions of convergence, and prove that, to a certain extent, among all the modified AOR methods the modified Gauss–Seidel method converges asymptotically faster. In section 4, a number of numerical examples show that the two methods of sections 2 and 3 are very effective and competitive with each other. Finally, section 5 contains a number of concluding remarks.

2. Scaled extrapolated modulus algorithms.

2.1. Introduction. To begin with, we transform the LCP using the "unknown" z and set

$$x = |z| + z \quad \text{and} \quad r = |z| - z, \tag{2.1}$$

whence from (1.1) we obtain the fixed-point equation

$$z = f(z) := D|z| + b, \quad D := (I + A)^{-1}(I - A), \quad b := -(I + A)^{-1}q \tag{2.2}$$

(see, e.g., [26]). First, we multiply through (1.1) by the scaling diagonal matrix $E := (\mathrm{diag}(A))^{-1}$ and then by Ω, a diagonal matrix with positive diagonal, which will be defined later. Using $\widehat{r} = \Omega E r$, $\widehat{A} = \Omega E A \Omega^{-1}$, $\widehat{x} = \Omega x$, $\widehat{q} = \Omega E q$ for r, A, x, q, respectively, we have the new LCP

$$\widehat{r} = \widehat{A}\widehat{x} + \widehat{q} \quad \text{and} \quad \widehat{r}^T\widehat{x} = 0, \quad \text{with} \quad \widehat{r} \geq 0 \;\; \text{and} \;\; \widehat{x} \geq 0. \tag{2.3}$$

Furthermore, using (2.1), with $\widehat{x}$, $\widehat{r}$, $\widehat{z}$ replacing x, r, z, respectively, in (2.3), we obtain that

$$\widehat{z} = f(\widehat{z}) := \widehat{D}|\widehat{z}| + \widehat{b}, \quad \widehat{D} := (I + \widehat{A})^{-1}(I - \widehat{A}), \quad \widehat{b} := -(I + \widehat{A})^{-1}\Omega E q. \tag{2.4}$$

Note that if $E = I$ and $\Omega = \omega I$, then $\widehat{D} = \omega I$, in (2.4), is the *extrapolated Cayley transform* of A [15], [17]. In this sense we can call $\widehat{D}$ the *scaled extrapolated Cayley transform* of A.

2.2. Scaling and extrapolating the LCP. To iteratively solve (2.2) we consider

$$\widehat{z}^{(m+1)} = \widehat{D}|\widehat{z}^{(m)}| + \widehat{b}, \quad m = 0, 1, 2, \ldots, \quad \text{with any } \widehat{z}^{(0)} \in \mathbb{R}^n. \tag{2.5}$$

Convergence of (2.5) to the unique solution of (2.4) is guaranteed if an *absolute matrix norm* of $\widehat{D}$ satisfies $||\widehat{D}|| < 1$. (For the properties of the *absolute matrix norm* induced by an *absolute vector norm*, see, e.g., [29], [21], or [26], and recall that the vector norms $||y||_p := (\sum_{i=1}^n |y_i|^p)^{\frac{1}{p}} \ \forall\, p \geq 1$ constitute *absolute vector norms*.) Since A in (1.1) is an H_+*-matrix*, so is EA, with $(EA)_{ii} = 1 \ \forall\, i = 1(1)n$. Also, $\widehat{A} = \Omega E A \Omega^{-1}$, with $(\widehat{A})_{ii} = 1 \ \forall\, i = 1(1)n$, is an H_+-matrix as it is diagonally similar to EA. If $\widehat{A}$ is, in addition, SDD, then an upper bound for $||\widehat{D}||_\infty (< 1)$ can be found. For this we use a slight modification of a lemma in [19] (see [20], [6]) presented below.

LEMMA 2.1. *If $\widehat{A} \in \mathbb{R}^{n,n}$ is an SDD matrix with diagonal elements $\widehat{a}_{ii} = 1 \ \forall\, i = 1(1)n$, then*

$$||\widehat{D}||_\infty = ||(I + \widehat{A})^{-1}(I - \widehat{A})||_\infty \leq \frac{||\widehat{B}||_\infty}{2 - ||\widehat{B}||_\infty} < 1, \quad \text{where} \quad \widehat{A} = I + \widehat{B}. \tag{2.6}$$

Proof. By assumption $||\widehat{B}||_\infty < 1$. Thus, using simple matrix norm properties we obtain

$$\begin{aligned} ||\widehat{D}||_\infty &= ||(I + \widehat{A})^{-1}(I - \widehat{A})||_\infty = ||(2I + \widehat{B})^{-1}\widehat{B}||_\infty \leq ||(2I + \widehat{B})^{-1}||_\infty ||\widehat{B}||_\infty \\ &= \tfrac{1}{2}\big\|\big(I + \tfrac{1}{2}\widehat{B}\big)^{-1}\big\|_\infty ||\widehat{B}||_\infty \leq \tfrac{1}{2\left(1 - \frac{1}{2}||\widehat{B}||_\infty\right)} ||\widehat{B}||_\infty = \tfrac{||\widehat{B}||_\infty}{2 - ||\widehat{B}||_\infty} < 1. \end{aligned}$$

∎

To exploit Lemma 2.1 we use a slight modification of Algorithm **H** (**AH**) [2], regardless of whether A is irreducible or reducible, to determine a diagonal matrix Ω, with $(\Omega)_{ii} > 0 \ \forall\, i = 1(1)n$, so that $\widehat{A} = \Omega E A \Omega^{-1}$ is SDD. For this, either we do as many iterations of **AH** as may be required to obtain an upper bound for $||\widehat{B}||_\infty = ||\widehat{A} - I||_\infty \ (< 1)$ or we go on iterating until two successive approximations of the upper bound coincide to a number of decimal places. Either way can be used since the bound in question decreases monotonically and tends to $\rho(|\widehat{B}|)$ [2] or [3]. Note also that $|\widehat{A}| = |\Omega E A \Omega^{-1}|$ and $\langle A \rangle$ share the same spectral radius for their associated Jacobi iteration matrices. Thus, deriving $\widehat{A}$, Lemma 2.1 applies and (2.5) converges. With the second choice our scheme converges asymptotically faster at the expense of more calculations per iteration. In the present work we use the first choice and keep Ω fixed, but E is recalculated.

2.3. Scaled extrapolated block modulus algorithm (SEBMA). In [18] two lemmas from [21] were cited, and three theorems were proved. Since we now work with $||\cdot||_\infty$ instead of $||\cdot||_2$, we cite the two lemmas as they are given in [21]. The proofs of the corresponding theorems will only be outlined since they are similar to those in [18].

Van Bokhoven's *modulus algorithm (MA).*

LEMMA 2.2. *Under the notation and the assumptions so far, if we apply Van Bokhoven's MA to scheme* (2.5)*, with* $\widehat{z}^{(0)} = 0 \in \mathbb{R}^n$*, then after* $\mathcal{N}$ *iterations,*

$$\mathcal{N} = \left\lceil \frac{\ln(\frac{1-c}{1+c}) - \ln(1+\frac{c_2}{c_1})}{\ln c} \right\rceil, \tag{2.7}$$

where $c := ||\widehat{D}||_\infty$*, and the component of maximum modulus of* $z^{(\mathcal{N})}$ *will become positive (or negative) and will remain positive (or negative) thereafter.*

Remark 2.1. The constants c_1 and c_2 in the case of the maximum norm ($||\cdot||_\infty$) are $c_1 = c_2 = 1$, and these values will be used in the corresponding expressions in what follows.

THEOREM 2.1. *Under the assumptions of Lemma* 2.2*,* $\mathcal{N}$ *is an increasing function of* $c := ||\widehat{D}||_\infty$.

Proof. Let $\widehat{\mathcal{N}} := \frac{\ln\left(\frac{1-c}{1+c}\right) - \ln 2}{\ln c}$ be the argument in the *ceiling* function in (2.7). It can be proved that $\frac{d\widehat{\mathcal{N}}}{dc} > 0$; therefore $\widehat{\mathcal{N}}$ strictly increases, and so $\mathcal{N}$ is an increasing function of c. ∎

Kappel and Watson's *block modulus algorithm (BMA).*

LEMMA 2.3. *If MA is applied to* (2.5)*, with* $\widehat{z}^{(0)} = 0 \in \mathbb{R}^n$*, then after* $\mathcal{N}$ *iterations, with* $\mathcal{N}$ *of* (2.7)*, all the components of* $\widehat{z}^{(\mathcal{N})}$ *satisfying*

$$|\widehat{z}_l^{(\mathcal{N})}| \geq T := \left(\frac{1}{1+c} - \frac{c^{\mathcal{N}}}{1-c}\right) ||\widehat{b}||_\infty \tag{2.8}$$

will preserve their sign thereafter.

It is clear that there may be more than one component of $\widehat{z}^{(\mathcal{N})}$ satisfying (2.8). It is then expected that *Kappel and Watson's algorithm (BMA)* will produce the solution sought in fewer iterations within a cycle, and in fewer cycles than *Van Bokhoven's (MA).*

THEOREM 2.2. *Under the assumptions of Lemma* 2.3*, considering* T *as a function of* c *only,* T *strictly decreases as* c *increases.*

Proof. From our assumptions $\frac{dT}{dc}$ and $\frac{d\widehat{T}}{dc}$, with $\widehat{T} := \frac{T}{||\widehat{b}||_\infty}$, are of the same sign. Working as in Theorem 3.2 of [18], $\frac{d\widehat{T}}{dc} < 0$ follows. Hence, $\widehat{T}$ and T are strictly decreasing functions of c. ∎

The contradiction in Theorems 2.1 and 2.2 seems to not hold in practice, and an explanation of this can be given by the following statement.

THEOREM 2.3. *As* $c \in (0,1)$ *decreases, the number* $\widehat{\mathcal{N}}$ *of Theorem* 2.1 *decreases, faster than the value* $\widehat{T}$ *of Theorem* 2.2 *increases. Specifically,* $\frac{d(\widehat{\mathcal{N}}\widehat{T})}{dc} > 0$.

Proof. Working as in Theorem 3.3 of [18], with c replacing ρ used there, we can find that $\frac{d(\widehat{\mathcal{N}}\widehat{T})}{dc} > 0$, which proves our claim. ∎

2.4. Further theoretical background. In this section we investigate the case of *scaled extrapolated modulus algorithm (SEMA).* The results obtained can be obviously generalized to cover the case of SEBMA.

Our method requires at most $n-1$ cycles to obtain the solution. However, as we go from one cycle of MA to the next, the system we work out is reduced in size and so we do fewer calculations per cycle. As in [18] the *scaled extrapolation* applied to each new LCP will be much faster than that applied to the old problem. To realize this we have to distinguish the two cases of the sign of the component of the maximum modulus of $\widehat{z}^{(\mathcal{N})}$, say $\widehat{z}_l^{(\mathcal{N})}$. If $\widehat{z}_l^{(\mathcal{N})} < 0$, then $\widehat{x}_l^{(\mathcal{N})} = 0$ and so the lth equation of the LCP and the lth column of $\widehat{A}$ are deleted. If $\widehat{z}_l^{(\mathcal{N})} > 0$, then $\widehat{r}_l^{(\mathcal{N})} = 0$, and so a Gauss elimination takes place with pivot $\widehat{a}_{ll}$ before the LCP is reduced in size as was done in the previous case. To simplify our analysis we assume that $l = 1$ and examine what happens to the ith row of the reduced matrix, $\forall\, i = 2(1)n$, after the first iteration of the first cycle. Generalization and induction provide the complete proofs.

THEOREM 2.4. *Let $\widehat{A} \in \mathbb{R}^{n,n}$ be an SDD matrix, with $\widehat{a}_{ii} = 1\ \forall\, i = 1(1)n$. The submatrix obtained by deleting the first row and column of $\widehat{A}$, $\widehat{A}'$ possesses the same properties.*

Proof. Obviously, $\widehat{a}'_{ii} = 1\ \forall\, i = 2(1)n$. By observing that $\sum_{j=2,\, j\neq i}^{n} |\widehat{a}'_{ij}| \equiv \sum_{j=2,\, j\neq i}^{n} |a_{ij}| \leq \sum_{j=1,\, j\neq i}^{n} |a_{ij}| < 1\ \forall\, i = 2(1)n$, the proof is completed. ∎

COROLLARY 2.1. *Under the assumptions of the theorem it is implied that*

$$\widehat{B}' = \widehat{A}' - I \Longrightarrow ||\widehat{B}'||_\infty \leq ||\widehat{B}||_\infty\ (< 1).$$

THEOREM 2.5. *Under the assumptions of Theorem* 2.4*, pivoting about the* $(1,1)$ *element of $\widehat{A}$ and deleting the first row and column of the resulting matrix, the submatrix $\widehat{A}'$ obtained is SDD with $\widehat{a}'_{ii}\ \forall\, i = 2(1)n$. Premultiplying it by $E_1 = (\mathrm{diag}(\widehat{A}'))^{-1} \in \mathbb{R}^{n-1,n-1}$, $\widehat{A}'' = E_1\widehat{A}'$ is SDD with $\widehat{a}''_{ii} = 1\ \forall\, i = 2(1)n$.*

Proof. After the described process takes place, the matrix $\widehat{A}'$ has elements given by

$$\widehat{a}'_{ij} = \widehat{a}_{ij} - \frac{\widehat{a}_{i1}}{\widehat{a}_{11}}\widehat{a}_{1j} = \widehat{a}_{ij} - \widehat{a}_{i1}\widehat{a}_{1j} \quad \forall\, i,j = 2(1)n, \tag{2.9}$$

since $\widehat{a}_{11} = 1$. It is $\widehat{a}'_{ii} > 0\ \forall\, i = 2(1)n$, because from (2.9) and the point just made we have

$$\widehat{a}'_{ii} = \widehat{a}_{ii} - \widehat{a}_{i1}\widehat{a}_{1i} = 1 - \widehat{a}_{i1}\widehat{a}_{1i} \quad \forall\, i = 2(1)n. \tag{2.10}$$

By Theorem 2.4 every principal submatrix of an SDD matrix is also SDD. So, $\begin{bmatrix} \widehat{a}_{11} & \widehat{a}_{1i} \\ \widehat{a}_{i1} & \widehat{a}_{ii} \end{bmatrix} = \begin{bmatrix} 1 & \widehat{a}_{1i} \\ \widehat{a}_{i1} & 1 \end{bmatrix}\ \forall\, i = 2(1)n$ has as its comparison matrix a nonsingular M-matrix. Hence it has a positive determinant $(1 - |\widehat{a}_{1i}||\widehat{a}_{i1}| > 0)$. Since $1 - \widehat{a}_{1i}\widehat{a}_{i1} \geq 1 - |\widehat{a}_{1i}||\widehat{a}_{i1}|\,(> 0)$ our claim is proved. To prove that $\widehat{A}'' = E_1\widehat{A}'$, which has $\widehat{a}''_{ii} = 1\ \forall\, i = 2(1)n$, is SDD, it is equivalent to prove that $\widehat{A}'$ is SDD. For this we can obtain

$$\begin{aligned} &\textstyle\sum_{j=2,\ j\neq i}^{n} \left|\widehat{a}'_{ij}\right| = \sum_{j=2,\ j\neq i}^{n} |\widehat{a}_{ij} - \widehat{a}_{i1}\widehat{a}_{1j}| \leq \sum_{j=1,\ j\neq i}^{n} |\widehat{a}_{ij}| \\ \Longleftarrow\quad &\textstyle\sum_{j=2,\ j\neq i}^{n} |\widehat{a}_{ij}| + \sum_{j=2,\ j\neq i}^{n} |\widehat{a}_{i1}|\,|\widehat{a}_{1j}| \leq |\widehat{a}_{i1}| + \sum_{j=2,\ j\neq i}^{n} |\widehat{a}_{ij}| \\ \Longleftrightarrow\quad &0 \leq |\widehat{a}_{i1}|\big(1 - \textstyle\sum_{j=2,\ j\neq i}^{n} |\widehat{a}_{ij}|\big) \quad \forall\, i = 2(1)n. \end{aligned} \tag{2.11}$$

The last relation is valid because $\widehat{A}$ is SDD and $\widehat{a}_{ii} = 1$. If any $\widehat{a}_{i1} \neq 0$, $i \in \{2, 3, \ldots, n\}$, the last inequality in (2.11) is strict and so is the first. ∎

COROLLARY 2.2. *Under the assumptions of the theorem, it is implied that*

$$\widehat{B}'' = \widehat{A}'' - I \Longrightarrow ||\widehat{B}''||_\infty \leq ||\widehat{B}||_\infty\ (< 1).$$

3. Modulus-based matrix splitting iteration methods.

3.1. Introduction. The most recent iterative methods for the solution of the LCP, when A is real positive definite or an H_+-matrix, are the *modulus-based matrix splitting iteration methods* introduced by Bai [6]. They are based on the substitutions

$$x = \frac{1}{2}(|z| + z) \quad \text{and} \quad r = \frac{1}{2}\Omega(|z| - z), \tag{3.1}$$

where Ω is a diagonal matrix with $\Omega_{ii} > 0 \,\forall\, i = 1(1)n$. Using the splitting $A = M - N$, LCP (1.1) is rewritten as

$$(\Omega + M)z = Nz + (\Omega - A)|z| - 2q. \tag{3.2}$$

Thus, the fixed-point equation (3.2) yields the iterative scheme

$$(\Omega + M)z^{(m+1)} = Nz^{(m)} + (\Omega - A)|z^{(m)}| - 2q,\ m = 0, 1, 2, \ldots, \quad \text{with any } z^{(0)} \in \mathbb{R}^n. \tag{3.3}$$

For the solution of the LCP by iterative method (3.3), in case A is an H_+-matrix, Theorems 4.3 and 4.4 are stated and proved in [6]. Below, we outline the results of Theorem 4.3 and give Theorem 4.4 in a condensed form. In both theorems, it is assumed that $A = M - N$ is an *H-compatible splitting* of A, i.e., $\langle A\rangle = \langle M\rangle - |N|$, which gives $\mathrm{diag}(M) = \mathrm{diag}(A) + \mathrm{diag}(N) \geq \mathrm{diag}(A)$. Then, under certain sufficient conditions, $z^{(m)}$ converges to the exact solution z_* of (3.2). Hence $\{x^{(m)}\}_{m=0}^{+\infty}$ converges to x_* the unique solution of the LCP in (1.1). From (3.3) and (3.2) it is obtained that

$$|z^{(m+1)} - z_*| \leq \widehat{L}_\Omega |z^{(m)} - z_*|,$$
$$\widehat{L}_\Omega = \widehat{M}_\Omega^{-1}\widehat{N}_\Omega, \quad \widehat{M}_\Omega = \Omega + \langle M\rangle,\ \widehat{N}_\Omega = |\Omega - M| + 2|N|,\ \widehat{A}_\Omega = \widehat{M}_\Omega - \widehat{N}_\Omega. \tag{3.4}$$

Since $\widehat{M}_\Omega - \widehat{N}_\Omega$ is an M-splitting, sufficient conditions for $\widehat{A}_\Omega$ to be a nonsingular M-matrix are (i) $\Omega \geq \frac{1}{2}\mathrm{diag}(M)$ and (ii) $\rho(I - (\mathrm{diag}(A))^{-1}\langle A\rangle) < \frac{1}{2}$, leading to $\rho(\widehat{L}_\Omega) < 1$ [6].

It is observed that condition $\rho(I - (\mathrm{diag}(A))^{-1}\langle A\rangle) < \frac{1}{2}$ makes *no* full exploitation of the H_+-character of A for which $\rho(I - (\mathrm{diag}(A))^{-1}\langle A\rangle) < 1$. Writing $\widehat{A}_\Omega$ as

$$\widehat{A}_\Omega = (\Omega - \mathrm{diag}(M) - |\Omega - \mathrm{diag}(M)|) + (2\mathrm{diag}(A) - 2\,(\mathrm{diag}(A) - \langle A\rangle))\,, \tag{3.5}$$

sufficient conditions for $\widehat{A}_\Omega$ to be a nonsingular M-matrix are (i) $\Omega - \mathrm{diag}(M) - |\Omega - \mathrm{diag}(M)| \geq O \Leftrightarrow \Omega \geq \mathrm{diag}(M)$, and (ii) the M-splitting $2\mathrm{diag}(A) - 2(\mathrm{diag}(A) - \langle A\rangle)$ is a nonsingular M-matrix $\Leftrightarrow \rho(I - (\mathrm{diag}(A))^{-1}\langle A\rangle) < 1$. Hence $\rho(\widehat{L}_\Omega) = \rho(I - (\mathrm{diag}(A))^{-1}\langle A\rangle) < 1$, and (3.3) converges. The new conditions lead to an improvement of all the relevant results in [6].

THEOREM 3.1 (Theorem 4.4 of [6]). *Let $A \in \mathbb{R}^{n,n}$ be an H_+-matrix with $A = D - L - U =: D - B$ satisfying $\rho := \rho(D^{-1}|B|) < \frac{1}{2}$, where $D = \mathrm{diag}(A)$ and L and U*

are strictly lower and strictly upper triangular matrices, respectively. Assume that Ω *is a diagonal matrix and* β *a real parameter satisfying* $\Omega \geq \frac{1}{2\alpha}D$, $\alpha > 0$, *and* $\beta \in (0, \alpha]$, *respectively. Then, the modified AOR (MAOR) iteration method, associated with the splitting*

$$A = M - N, \quad M = \frac{1}{\alpha}(D - \beta L), \ N = \frac{1}{\alpha}[(1-\alpha)D + (\alpha - \beta)L + \alpha U], \tag{3.6}$$

converges for any initial vector and any

$$\beta \in (0, \alpha] \ \ \textit{and} \ \ \alpha \in \left(\frac{1}{2(1-\rho)}, \frac{3}{2(1+\rho)}\right). \tag{3.7}$$

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1 *we have the following: For* $\alpha = \beta$ *the modified SOR (MSOR) method converges for the same values of* α *as in* (3.7). *The modified Gauss–Seidel (MGS) method,* $\beta = \alpha = 1$, *and the modified Jacobi (MJ) method,* $\beta = 0$, $\alpha = 1$, *converge.*

3.2. Improved Bai's MAOR method. In Theorem 3.1, A is an H_+-matrix, so it would be more natural to require that the MAOR method converge for any $\rho < 1$ instead of for $\rho < \frac{1}{2}$. For this (3.4) and (3.6) give

$$\begin{gathered} \widehat{M}_\Omega = \Omega + \langle M \rangle = \Omega + \tfrac{1}{\alpha}D - \tfrac{\beta}{\alpha}|L|, \\ \widehat{N}_\Omega = |\Omega - M| + 2|N| = |\Omega - \tfrac{1}{\alpha}D| + \tfrac{\beta}{\alpha}|L| + \tfrac{2|1-\alpha|}{\alpha}D + \tfrac{2|\alpha-\beta|}{\alpha}|L| + 2|U|, \\ \widehat{A}_\Omega = \widehat{M}_\Omega - \widehat{N}_\Omega = \Omega - |\Omega - \tfrac{1}{\alpha}D| + \tfrac{1-2|1-\alpha|}{\alpha}D - 2|B|. \end{gathered} \tag{3.8}$$

"Restricting" the assumptions on Ω to $\Omega \geq \frac{1}{\alpha}D$ and $\beta \in [0, \alpha]$, $\alpha > 0$, $\widehat{A}_\Omega$ is simplified to $\widehat{A}_\Omega = \frac{2(1-|1-\alpha|)}{\alpha}D - 2|B|$. So, $\widehat{A}_\Omega$ is an M-matrix iff $1 - |1-\alpha| > 0$ and $\frac{\alpha}{1-|1-\alpha|}\rho\left(D^{-1}|B|\right) < 1$. However, the first inequality holds iff $\alpha \in (0, 2)$, and the second one holds iff $\max_{\alpha \in (0,2)}\{\frac{\alpha}{1-|1-\alpha|}\} < \frac{1}{\rho} \Leftrightarrow \alpha \in (0, \frac{2}{1+\rho})$. Hence, an improved form of Theorem 4.4 of [6] (Theorem 3.1 above) can be given.

THEOREM 3.2. *Let* $A \in \mathbb{R}^{n,n}$ *be an* H_+*-matrix with* $A = D - L - U =: D - B$, *where* $D = \operatorname{diag}(A)$, L *and* U *are strictly lower and strictly upper triangular matrices, respectively, and* $\rho(B) > 0$. *Assume that* Ω *is a diagonal matrix and* β *a real parameter satisfying* $\Omega \geq \frac{1}{\alpha}D$, $\alpha > 0$, *and* $\beta \geq 0$. *Then, the improved MAOR corresponding to splitting* (3.6) *converges for any initial vector and any*

(3.9)

$$\alpha \in \left(0, \tfrac{2}{1+\rho}\right) \ \textit{and} \ \left\{ \begin{array}{c} \beta \in [0, \alpha] \\ \textit{and also} \\ \beta \in \left[\alpha, \frac{1}{2}(\gamma_\alpha + 1)\alpha\right), \\ \textit{where } \gamma_\alpha \textit{ is such that} \\ \rho\left(D^{-1}(\gamma_\alpha|L| + |U|)\right) = \frac{1-|1-\alpha|}{\alpha} = \left\{ \begin{array}{ll} 1 & \textit{for } \alpha \in (0, 1], \\ \frac{2-\alpha}{\alpha} & \textit{for } \alpha \in \left[1, \frac{2}{1+\rho}\right). \end{array} \right. \end{array} \right.$$

Note. We assume $\rho(|B|) > 0$ for otherwise A, or a similarity permutation of it, would be an upper triangular matrix (*Frobenius normal form*), and the solution to the LCP could be found by a back-substitution–type process.

Proof. Apparently, for $\beta \in [0, \alpha]$, the interval of convergence for α is that found in the analysis that precedes the present Theorem 3.2. So, the only case we examine

is that of $\beta \geq \alpha$. In this case,

$$\widehat{A}_\Omega = \widehat{M}_\Omega - \widehat{N}_\Omega = \frac{2(1-|1-\alpha|)}{\alpha} D - 2(\gamma|L| + |U|), \quad \gamma := \frac{2\beta}{\alpha} - 1. \tag{3.10}$$

$\widehat{A}_\Omega$ is an M-matrix iff $1-|1-\alpha| > 0$ and $\max_{0<\alpha\leq\beta}\{\frac{\alpha}{(1-|1-\alpha|)}\rho(D^{-1}(\gamma|L|+|U|))\} < 1$, $\gamma = \frac{2\beta}{\alpha} - 1$. The first relation is equivalent to $\alpha \in (0,2)$. For a fixed α, the second relation is written as

$$\max_{0<\alpha\leq\beta} \left\{\rho\left(D^{-1}(\gamma|L| + |U|)\right)\right\} < \frac{1-|1-\alpha|}{\alpha}, \quad \gamma = \frac{2\beta}{\alpha} - 1 \geq 1. \tag{3.11}$$

Due to $\rho(|B|) > 0$ and the nonnegativity of $\gamma|L| + |U|$, the *Perron–Frobenius theory* (see Varga [31]) suggests that as γ increases in $[1, +\infty)$, $\rho\left(D^{-1}\left(\gamma|L| + |U|\right)\right)$ strictly increases in $[\rho, +\infty)$ (see also the *Stein–Rosenberg theorem* in [31]). In addition, $\frac{1-|1-\alpha|}{\alpha}$ in (3.11) is 1 $\forall\ \alpha \in (0,1]$ and equals $\frac{2-\alpha}{\alpha}$ $\forall\ \alpha \in [1,2)$, with maximum 1 at $\alpha = 1$. In the former case there exists a $\gamma = \gamma_\alpha$ such that $\rho\left(D^{-1}(\gamma_\alpha|L| + |U|)\right) = 1$. Therefore, $1 \leq \gamma < \gamma_\alpha \Leftrightarrow \alpha \leq \beta < \frac{1}{2}(\gamma_\alpha + 1)\alpha$. In the latter case, for any fixed $\alpha \in [1,2)$ we must have $\max_{1\leq\alpha<2,\, \alpha\leq\beta} \rho\left(D^{-1}\left(\gamma|L| + |U|\right)\right) < \frac{2-\alpha}{\alpha}$. Since $\gamma \geq 1$, $\min_{\gamma\in[1,+\infty)} \rho\left(D^{-1}(\gamma|L| + |U|)\right) = \rho < \frac{2-\alpha}{\alpha}$, implying that $\alpha < \frac{2}{1+\rho}$. $\frac{2-\alpha}{\alpha}$ is a strictly decreasing function of $\alpha \in [1, \frac{2}{1+\rho})$ taking values from 1, included, to ρ, excluded. For $\gamma = 1$ it is $\rho\left(D^{-1}(1.|L| + |U|)\right) = \rho$. So, for $\gamma \in [1, \gamma_\alpha)$, where γ_α satisfies $\rho\left(D^{-1}(\gamma_\alpha|L| + |U|)\right) = \frac{2-\alpha}{\alpha}$, $\beta \in [\alpha, \frac{1}{2}(\gamma_\alpha + 1)\alpha)$.

Collecting the results of the preceding analysis, the second set of restrictions in (3.9) holds. ∎

Remark 3.1. Note that when the original version of this paper was written, a paper by Zhang [33] came to the first author's attention. Based on a two-step consideration of Bai's method, Zhang succeeded in fully exploiting the fact that $\rho\left(D^{-1}|B|\right) < 1$ and found that the two-step MAOR converges $\forall\Omega = \mathrm{diag}(M)$ and $0 < \beta \leq \alpha < \frac{2}{1+\rho}$.

3.3. "Best" improved MAOR method. In this section an effort is made to determine the "best" possible improved MAOR by making its associated $\rho(\widehat{L}_\Omega)$, of $\widehat{L}_\Omega$ in (3.4), as small as possible. We assume that $\rho < 1$, $\Omega \geq \frac{1}{\alpha}D$, and (α, β) are given by (3.9). First, we prove that the improved MAOR of Theorem 3.2 is "better" than that of Theorem 3.1. Next, we show that for a fixed $\alpha \in (0, \frac{2}{1+\rho})$ the "best" improved MAOR is the improved MSOR. Finally, we prove that the improved MGS is the "best" of all the improved MSORs. Our analysis is based on the following statement.

THEOREM 3.3 (Theorem 3.13 of Marek and Szyld in [25]). *Let $A_i = M_i - N_i$ be weak nonnegative splittings with $L_i := M_i^{-1}N_i$ and $\rho(L_i) < 1$, $i = 1,2$. Let $x_i \geq 0$ be such that $L_i x_i = \rho(L_i)x_i$, $i = 1,2$. Let $A_2^{-1} \geq O$ and $A_2^{-1} \geq A_1^{-1}$. If either $N_2x_1 \geq N_1x_1 \geq 0$ or $N_2x_2 \geq N_1x_2 \geq 0$, with $x_2 > 0$, then $\rho(L_1) \leq \rho(L_2)$. Moreover, if $A_2^{-1} > O$ and $N_1 \neq N_2$, then $\rho(L_1) < \rho(L_2)$.*

Note: In Theorem 3.13 in [25], the operators L_i, $i = 1,2$, have *property "d."* As explained, the theorem refers to a *Banach real space* $\mathcal{E}$, generated by a *normal closed cone* K, and to all *bounded linear operators* $\mathcal{B}(\mathcal{E})$ mapping $\mathcal{E}$ into itself. When $\mathcal{E} := \mathbb{R}^n$, a generating cone is the set of all nonnegative vectors; then all nonnegative matrices in $\mathbb{R}^{n,n}$ have property "*d*."

THEOREM 3.4. *The improved MAOR method of Theorem* 3.2*, for any $\Omega \geq \frac{1}{\alpha}D$ and any fixed pair (α, β) satisfying* (3.7)*, is at least as good as the MAOR iterative method of Theorem* 3.1 *(Theorem* 4.4 *of* [6]*) with a fixed $\Omega = \frac{1}{2\alpha}D$.*

Proof. From (3.8) we have

$$\widehat{M}_{\Omega,1} = \frac{2}{\alpha}D - \frac{\beta}{\alpha}|L|,\ \widehat{N}_{\Omega,1} = \frac{2|1-\alpha|}{\alpha}D - \frac{\beta}{\alpha}|L| + 2|B|, \tag{3.12}$$
$$\widehat{A}_{\Omega,1} = \frac{2(1-|1-\alpha|)}{\alpha}D - 2|B|,$$
$$\widehat{M}_{\Omega,2} = \frac{3}{2\alpha}D - \frac{\beta}{\alpha}|L|,\ \widehat{N}_{\Omega,2} = \frac{(1+4|1-\alpha|)}{2\alpha}D - \frac{\beta}{\alpha}|L| + 2|B|, \tag{3.13}$$
$$\widehat{A}_{\Omega,2} = \frac{(1-2|1-\alpha|)}{\alpha}D - 2|B|,$$

respectively. As has already been proved, $\widehat{A}_{\Omega,i} = \widehat{M}_{\Omega,i} - \widehat{N}_{\Omega,i}$, $i = 1, 2$, are M-splittings of nonsingular M-matrices, with $\widehat{N}_{\Omega,2} \geq \widehat{N}_{\Omega,1} \geq O$ and $\widehat{N}_{\Omega,2} \neq \widehat{N}_{\Omega,1}$. Hence, for $\widehat{L}_{\Omega,i} = (\widehat{M}_{\Omega,i})^{-1}\widehat{N}_{\Omega,i}$, $\rho_i = \rho(\widehat{L}_{\Omega,i}) < 1$, where $\widehat{L}_{\Omega,i}x_i = \rho_i x_i$, with ρ_i the spectral radii and $x_i \geq 0$ their associated eigenvectors of $\widehat{L}_{\Omega,i}$, $i = 1, 2$. Obviously, $(\widehat{A}_{\Omega,i})^{-1} \geq O$, $i = 1, 2$. Then let $\delta_1 := \frac{\alpha}{2(1-|1-\alpha|)} < \frac{\alpha}{1-2|1-\alpha|} =: \delta_2 < \frac{1}{\rho}$ because the first inequality always holds, while the second one holds because of the restrictions on $\rho < \frac{1}{2}$ and α in (3.7). Since $\rho(2\delta_i D^{-1}B) < 1$, $i = 1, 2$, we use Neumann expansions to obtain

$$\begin{aligned}(\widehat{A}_{\Omega,i})^{-1} &= \left(\tfrac{1}{\delta_i}D - 2|B|\right)^{-1} = \delta_i \left(I - 2\delta_i D^{-1}|B|\right)^{-1} D^{-1} \\ &= \delta_i \left(I + (2\delta_i D^{-1}|B|) + (2\delta_i D^{-1}|B|)^2 + \cdots\right) D^{-1} \ (\geq O).\end{aligned}$$

Consequently, $(\widehat{A}_{\Omega,2})^{-1} \geq (\widehat{A}_{\Omega,1})^{-1} \geq O$. Also, $\widehat{N}_{\Omega,2}x_1 \geq \widehat{N}_{\Omega,1}x_1 \geq 0$. Therefore, the first part of the requirements of Theorem 3.3 is fulfilled; hence $\rho(\widehat{L}_{\Omega,1}) \leq \rho(\widehat{L}_{\Omega,2})$, and our statement holds. Note that if $(\widehat{A}_{\Omega,2})^{-1} > O$, as, e.g., when A is irreducible, then so are $\widehat{A}_\Omega$ and $\widehat{A}_{\Omega,i}$, $i = 1, 2$, and because $\widehat{N}_{\Omega,2} \neq \widehat{N}_{\Omega,1}$, the inequality regarding the spectral radii is strict ($\rho(\widehat{L}_{\Omega,1}) < \rho(\widehat{L}_{\Omega,2})$). ∎

THEOREM 3.5. *Let $\alpha \in (0, \frac{2}{1+\rho})$ in (3.9) be fixed, and let any $\Omega \geq \frac{1}{\alpha}D$. Then, the spectral radius of the MAOR method associated with the splitting of Theorem 3.2 is a decreasing function of $\beta \in (0, \alpha]$. Therefore, the "best" MAOR method of Theorem 3.2 is the MSOR method.*

Proof. For $0 \leq \beta_2 < \beta_1 \leq \alpha$, the MAOR methods corresponding to the two β's are

$$\widehat{M}_{\Omega,i} = \frac{2}{\alpha}D - \frac{\beta_i}{\alpha}|L|,\ \widehat{N}_{\Omega,i} = \frac{2|1-\alpha|}{\alpha}D - \frac{\beta_i}{\alpha}|L| + 2|B|,\ \widehat{A}_{\Omega,i} = \frac{2(1-|1-\alpha|)}{\alpha}D - 2|B|$$

for $i = 1, 2$. Note that $\widehat{A}_{\Omega,2} \equiv \widehat{A}_{\Omega,1}$ and the proof can be based on Theorem 3.5 in [25], a simplified version of Theorem 3.3 for $\widehat{A}_\Omega \equiv \widehat{A}_{\Omega,2} = \widehat{A}_{\Omega,1}$. Since the proof is much easier, it suffices to note that $(\widehat{A}_\Omega)^{-1} \geq O$, $\widehat{N}_{\Omega,2} \geq \widehat{N}_{\Omega,1} \geq O$, with $\widehat{N}_{\Omega,2} \neq \widehat{N}_{\Omega,1}$. Then, $\rho(\widehat{L}_{\Omega,1}) \leq \rho(\widehat{L}_{\Omega,2})$, and the conclusion immediately follows. For $\widehat{A}_\Omega > O$, $\rho(\widehat{L}_{\Omega,1}) < \rho(\widehat{L}_{\Omega,2})$. ∎

THEOREM 3.6. *Under the assumptions of Theorem 3.5, the spectral radius of the MAOR associated with the splitting of Theorem 3.2 is an increasing function of $\beta \in \left[\alpha, \frac{1}{2}(\gamma_\alpha + 1)\alpha\right)$. Hence, the "best" MAOR of Theorem 3.2 is the MAOR for $\beta \to \alpha^+ \in \left[\alpha, \frac{1}{2}(\gamma_\alpha + 1)\alpha\right)$.*

Proof. For $\alpha < \beta_1 < \beta_2 < \frac{1}{2}(\gamma_\alpha + 1)$ the MAOR methods corresponding to β_1 and β_2 are

$$\begin{aligned} &\widehat{M}_{\Omega,i} = \tfrac{2}{\alpha}D - \tfrac{\beta_i}{\alpha}|L|, \quad \widehat{N}_{\Omega,i} = \tfrac{2|1-\alpha|}{\alpha}D + (\tfrac{3\beta_i}{\alpha} - 2)|L| + 2|U|, \\ &\widehat{A}_{\Omega,i} = \tfrac{2(1-|1-\alpha|)}{\alpha}D - 2(\gamma_i|L| + |U|), \ \gamma_i := \tfrac{2\beta_i}{\alpha} - 1\,(\geq 1), \ i = 1, 2. \end{aligned} \tag{3.14}$$

Observe that $1 < \gamma_1 < \gamma_2 < \gamma_\alpha \ \forall \alpha \in (0, \frac{2}{1+\rho})$. Working as in the proof of Theorem 3.4, we find out that the splittings $\widehat{M}_{\Omega,i} - \widehat{N}_{\Omega,i}$, $i = 1, 2$, are M-splittings of nonsingular M-matrices, leading to the convergence of the associated iterative methods, with $\widehat{L}_{\Omega,i}$, $i = 1, 2$, being the corresponding iteration matrices. Using Neumann expansions we find that $(\widehat{A}_{\Omega,2})^{-1} \geq (\widehat{A}_{\Omega,1})^{-1} \geq O$. It is verified that $\widehat{N}_{\Omega,2} \geq \widehat{N}_{\Omega,1} \geq O$, with $\widehat{N}_{\Omega,2} \neq \widehat{N}_{\Omega,1}$. Since $\widehat{N}_{\Omega,2}x_1 \geq \widehat{N}_{\Omega,1}x_1$, where $x_i \geq 0$ is the eigenvector associated with the spectral radius ρ_i of $\widehat{L}_{\Omega,i}$, $i = 1, 2$, we conclude that $\rho(\widehat{L}_{\Omega,1}) \leq \rho(\widehat{L}_{\Omega,2})$. Note that if $(\widehat{A}_{\Omega,2})^{-1} > O$, the last inequality is strict. To conclude, observe that $\beta \to \alpha^+ \in \big[\alpha, \frac{1}{2}(\gamma_\alpha + 1)\alpha\big)$, $\gamma \to 1^+$, and the MAOR tends to the MSOR. Specifically,

$$\begin{aligned} &\inf_{\beta \to \alpha^+ \in \left[\alpha, \frac{1}{2}(\gamma_\alpha+1)\alpha\right)} \left(\left(\tfrac{2}{\alpha}D - \tfrac{\beta}{\alpha}|L|\right)^{-1} \left(\tfrac{2|1-\alpha|}{\alpha}D + (\tfrac{3\beta}{\alpha} - 2)|L| + 2|U|\right)\right) \\ &= \left(\tfrac{2}{\alpha}D - |L|\right)^{-1} (\tfrac{2|1-\alpha|}{\alpha}D + |L| + 2|U|). \end{aligned} \tag{3.15}$$

Note that the *infimum* is the MSOR iteration matrix associated with any $\beta = \alpha \in (0, \frac{2}{1+\rho})$. ∎

THEOREM 3.7. *Under the assumptions of Theorem* 3.2*, for a fixed* α *satisfying* (3.9) *and any* $\Omega \geq \frac{1}{\alpha}D$*, the "best" MSOR is the MGS.*

Proof. Theorem 3.5 in combination with Theorem 3.6, applying continuity arguments to the latter, proves that the "best" MAOR is the MSOR. For $\beta = \alpha = 1$ and $\beta = \alpha \neq 1$ we get

$$\begin{aligned} &\widehat{M}_{\Omega,1} = 2D - |L|, \quad \widehat{N}_{\Omega,1} = |L| + 2|U|, \quad \widehat{A}_{\Omega,1} = 2D - 2|B|, \\ &\widehat{M}_{\Omega,2} = \tfrac{2}{\alpha}D - |L|, \quad \widehat{N}_{\Omega,2} = \tfrac{2|1-\alpha|}{\alpha}D + |L| + 2|U|, \quad \widehat{A}_{\Omega,2} = \tfrac{2(1-|1-\alpha|)}{\alpha}D - 2|B|. \end{aligned} \tag{3.16}$$

The two splittings in (3.16) are M-splittings of nonsingular M-matrices, with iterations matrices $\widehat{L}_{\Omega,i} = (\widehat{M}_{\Omega,i})^{-1}\widehat{N}_{\Omega,i} \geq O$, $i = 1, 2$, and spectral radii $\rho(\widehat{L}_{\Omega,i}) < 1$, $i = 1, 2$, having associated eigenvectors $x_i \geq 0$, $i = 1, 2$. Note that $(\widehat{A}_{\Omega,i})^{-1} \geq O$, $\widehat{N}_{\Omega,2} \geq \widehat{N}_{\Omega,1} \geq O$, $\widehat{N}_{\Omega,2} \neq \widehat{N}_{\Omega,1}$, and $\widehat{N}_{\Omega,2}x_1 \geq \widehat{N}_{\Omega,1}x_1$. Setting $\delta := \frac{\alpha}{(1-|1-\alpha|)}\,(\geq 1)$ and using Neumann expansions, it is proved that

$$\left(\widehat{A}_{\Omega,2}\right)^{-1} = \frac{\delta}{2}\left(I - \delta D^{-1}|B|\right)^{-1} D^{-1} \geq \frac{1}{2}\left(I - D^{-1}|B|\right)^{-1} D^{-1} = \left(\widehat{A}_{\Omega,1}\right)^{-1} \geq O.$$

Thus, by Theorem 3.3, $\rho(\widehat{L}_{\Omega,1}) \leq \rho(\widehat{L}_{\Omega,2})$, while for $(\widehat{A}_{\Omega,2})^{-1} > O$, it holds that $\rho(\widehat{L}_{\Omega,1}) < \rho(\widehat{L}_{\Omega,2})$. ∎

4. Numerical examples. From section 2, it becomes clear that if A in LCP (1.1) is an H_+-matrix, SEBMA is the best method of all three examined there. From section 3, the improved MGS, with $\Omega = \mathrm{diag}(M)$, is the best among all the improved MAOR methods of Theorem 3.2 and is better than the MAOR of [6], with

$\Omega = \frac{1}{2}\text{diag}(M)$. Since there is *no* apparent direct theoretical comparison between the best method of section 2 and the best method of section 3, we have to compare them by using Examples 5.1 and 5.2 of [6].

In our computations $\widehat{x}^{(0)} = x^{(0)} = [0,0,0,0,\ldots]^T \in \mathbb{R}^{m^2}$, $m^2 = n$, implying $\widehat{z}^{(0)} = z^{(0)} = [0,0,\ldots,0]^T \in \mathbb{R}^{m^2}$, so that SEBMA applies. Our runs were performed using MATLAB 7.5.0.

Since SEBMA is a semidirect method, it gives results with *no* stopping criteria imposed. It was observed that in all the runs of the two examples the quantity $||\overline{x} - x_*||_\infty$ ($\overline{x}$ is the computed solution and x_* the exact one) was of order 10^{-15}–10^{-16}. The corresponding absolute error was used as a stopping criterion for the improved MGS.

Note that the vectors x_* and r_* were given for each problem, while q was taken to be $q = r_* - Ax_*$.

Example 1 (Example 5.1 of [6]). Let LCP (1.1) with

$$A = \widetilde{A} + \mu I_n \in \mathbb{R}^{n,n}, \quad \widetilde{A} = \text{tridiag}(-I_m, S, -I_m) \in \mathbb{R}^{n,n}, \tag{4.1}$$
$$S = \text{tridiag}(-1,4,-1) \in \mathbb{R}^{m,m},$$
$$\mu = 4, \quad x_* = [1,2,1,2,\ldots]^T \in \mathbb{R}^n, \quad r_* = [0,0,0,0,\ldots]^T \in \mathbb{R}^n.$$

The results obtained are illustrated in Table 1, from which we can observe that, as was expected, when comparing the MGS and the improved MGS, the latter method performs better than the former. Also, it seems that for small n, both the MGS and the improved MGS outperform SEBMA, while as n becomes larger, the situation is reversed. Note that in the specific examples SEBMA needed only one cycle to find $\overline{x}$, and the number of iterations indicated are the inner ones within this cycle.

TABLE 1
Number of iterations (iter) and CPU times in seconds.

Results of methods for Example 1 ($\mu = 4$)					
n	$\rho(D^{-1}\|B\|)$		SEBMA	MGS ($\Omega = \frac{1}{2}\text{diag}(A)$)	Improved MGS ($\Omega = \text{diag}(A)$)
25	0.433	iter	3	70	36
		CPU	0.046	0.031	0.015
100	0.479	iter	3	80	41
		CPU	0.046	0.031	0.015
225	0.490	iter	3	92	42
		CPU	0.078	0.062	0.031
400	0.494	iter	3	88	42
		CPU	0.148	0.265	0.156
625	0.496	iter	3	98	42
		CPU	0.265	0.780	0.343

Example 2 (Example 5.2 of [6]). This differs from Example 1 in the nonsymmetric off-diagonal elements of A:

$$\widetilde{A} = \text{tridiag}(-1.5I_m, S, -0.5I_m) \in \mathbb{R}^{n,n}, \quad S = \text{tridiag}(-1.5, 4, -0.5) \in \mathbb{R}^{m,m}. \tag{4.2}$$

The results obtained are illustrated in Table 2. The points we can make from Table 2 are more or less the same as those for Example 1.

Example 3. This is Example 1 with $\mu = 2$. The results obtained are seen in Table 3, from which we can make the same points as those in Examples 1 and 2. Note

TABLE 2
Number of iterations (iter) and CPU times in seconds.

Results of methods for Example 2 ($\mu = 4$)					
n	$\rho(D^{-1}\|B\|)$		SEBMA	MGS ($\Omega = \frac{1}{2}\text{diag}(A)$)	Improved MGS ($\Omega = \text{diag}(A)$)
25	0.375	iter	3	61	31
		CPU	0.062	0.031	0.000
100	0.415	iter	3	58	36
		CPU	0.062	0.031	0.015
225	0.424	iter	3	67	37
		CPU	0.078	0.078	0.031
400	0.428	iter	3	59	38
		CPU	0.109	0.187	0.140
625	0.429	iter	3	67	38
		CPU	0.234	0.546	0.296

TABLE 3
Number of iterations (iter) and CPU times in seconds.

Results of methods for Example 3 ($\mu = 2$)					
n	$\rho(D^{-1}\|B\|)$		SEBMA	MGS ($\Omega = \frac{1}{2}\text{diag}(A)$)	Improved MGS ($\Omega = \text{diag}(A)$)
25	0.577	iter	3	86	51
		CPU	0.046	0.000	0.000
100	0.639	iter	3	104	62
		CPU	0.093	0.015	0.000
225	0.653	iter	3	123	66
		CPU	0.093	0.046	0.032
400	0.659	iter	3	129	67
		CPU	0.109	0.405	0.214
625	0.659	iter	3	145	68
		CPU	0.249	1.123	0.546

that $\mu = 2$ violates the condition of Theorem 3.1 ($\rho(D^{-1}|B| < \frac{1}{2}$)). Despite this, the MGS works. This should *not* come as a surprise since the analysis in the introduction of section 3 and also Theorem 4.3 of [6] give sufficient conditions for the convergence of the method and *not* necessary ones.

Example 4. This is Example 2 with $\mu = 2$. The results obtained are illustrated in Table 4, from which one can make the same points as those in Example 3, and are therefore omitted.

5. Concluding remarks. Before we conclude our work we would like to make a number of points:

(i) The theory developed in the present work was confirmed by numerous experiments. However, we should mention that when we were experimenting by changing the values of α and β, the optimal numerical values for them were very close to if not equal to 1.

(ii) As noted in Examples 1–4, it becomes clear that for small values of n, the improved MGS, and even MGS, outperforms SEBMA, while for larger values of n, the situation is reversed.

(iii) Regarding SEBMA, one may observe, experimentally, that whenever diag(A) is not a multiple of the unit matrix, as is the case in the presented examples, and A is an H_+-matrix but *not* SDD, the application of the modification of **AH** may be time consuming; then the improved MGS can be proved better even for larger n.

TABLE 4
Number of iterations (iter) and CPU times in seconds.

Results of methods for Example 4 ($\mu = 2$)					
n	$\rho(D^{-1}\|B\|)$		SEBMA	MGS ($\Omega = \frac{1}{2}\text{diag}(A)$)	Improved MGS ($\Omega = \text{diag}(A)$)
25	0.500	iter	3	73	42
		CPU	0.078	0.000	0.000
100	0.553	iter	3	72	51
		CPU	0.093	0.015	0.000
225	0.566	iter	3	83	55
		CPU	0.093	0.062	0.015
400	0.570	iter	3	73	58
		CPU	0.140	0.234	0.202
625	0.573	iter	3	83	59
		CPU	0.265	0.686	0.468

In such cases it would be advisable to use SEBMA only when many LCPs with the same matrix A but different q's are to be solved, since the modified **AH** should be applied only once (a situation analogous to the solution of linear systems of algebraic equations with more than one right-hand side).

(iv) Regarding the improved MAOR method of section 3, recall that the theoretical results obtained were under sufficient conditions. This seems to be unavoidable even in the iterative solution of large linear systems with an H-matrix, as, e.g., when the SOR method with parameter ω is to be applied. It is then well known that a sufficient condition for convergence is $\omega \in (0, \frac{2}{1+\rho(|D^{-1}||B|)})$ (see [30]). This is precisely the same as the condition found in (3.9) of Theorem 3.1 for the improved MSOR. Thus, an improvement of the theory of section 3 may be in the direction of determining a nonnegative operator that has a smaller spectral radius than that of $\widehat{L}_\Omega$ in the first relation of (3.4).

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REFERENCES

[1] B. H. AHN, *Solution of nonsymmetric linear complementarity problems by iterative methods*, J. Optim. Theory Appl., 33 (1981), pp. 175–185.
[2] M. ALANELLI AND A. HADJIDIMOS, *A new iterative criterion for H-matrices*, SIAM J. Matrix Anal. Appl., 29 (2006), pp. 160–176.
[3] M. ALANELLI AND A. HADJIDIMOS, *A new iterative criterion for H-matrices: The reducible case*, Linear Algebra Appl., 428 (2008), pp. 2761–2777.
[4] Z.-Z. BAI, *The convergence of parallel iteration algorithms for linear complementarity problems*, Comput. Math. Appl., 32 (1996), pp. 1–17.
[5] Z.-Z. BAI, *On the convergence of the multisplitting methods for the linear complementarity problem*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 67–78.
[6] Z.-Z. BAI, *Modulus-based matrix splitting iteration methods for linear complementarity problems*, Numer. Linear Algebra Appl., 17 (2010), pp. 917–933.
[7] Z.-Z. BAI AND D. J. EVANS, *Matrix multisplitting relaxation methods for linear complementarity problems*, Int. J. Comput. Math., 63 (1997), pp. 309–326.
[8] Z.-Z. BAI AND D. J. EVANS, *Matrix multisplitting methods with applications to linear complementarity problems: Parallel asynchronous methods*, Int. J. Comput. Math., 79 (2002), pp. 205–232.
[9] Z.-Z. BAI AND Y.-G. HUANG, *A class of asynchronous parallel multisplitting relaxation methods for the large sparse linear complementarity problems*, J. Comput. Math., 21 (2003), pp. 773–790.

[10] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics Appl. Math. 9, SIAM, Philadelphia, 1994.
[11] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, Academic Press, New York, 1992.
[12] C. W. Cryer, *The solution of a quadratic programming problem using systematic overrelaxation*, SIAM J. Control, 9 (1971), pp. 385–392.
[13] L. Cvetković and S. Rapajić, *How to improve MAOR method convergence area for linear complementarity problems*, Appl. Math. Comput., 162 (2005), pp. 577–584.
[14] J.-L. Dong and M.-Q. Jiang, *A modified modulus method for symmetric positive-definite linear complementarity problems*, Numer. Linear Algebra Appl., 16 (2009), pp. 129–143.
[15] S. M. Fallatt and M. J. Tsatsomeros, *On the Cayley transform of positivity classes of matrices*, Electron. J. Linear Algebra, 9 (2002), pp. 190–196.
[16] A. Frommer and D. B. Szyld, *H-splittings and two-stage iterative methods*, Numer. Math., 63 (1992), pp. 345–356.
[17] A. Hadjidimos and M. Tzoumas, *On the principle of extrapolation and the Cayley transform*, Linear Algebra Appl., 428 (2008), pp. 2761–2777.
[18] A. Hadjidimos and M. Tzoumas, *Nonstationary extrapolated modulus algorithms for the solution of the linear complementarity problem*, Linear Algebra Appl., 431 (2009), pp. 197–210.
[19] J.-G. Hu, *Estimates of $||B^{-1}A||_\infty$ and their applications*, Math. Numer. Sinica, 4 (1982), pp. 272–282 (in Chinese).
[20] J.-G. Hu, *Scaling transformation and convergence of splittings of matrix*, Math. Numer. Sin., 5 (1983), pp. 465–485 (in Chinese).
[21] N. W. Kappel and L. T. Watson, *Iterative algorithms for the linear complementarity problems*, Int. J. Comput. Math., 19 (1986), pp. 273–297.
[22] R. B. Kellogg, *On complex eigenvalues of M and P matrices*, Numer. Math., 19 (1972), pp. 170–175.
[23] M. D. Koulisianis and T. S. Papatheodorou, *Improving projected successive overrelaxation method for linear complementarity problems*, Appl. Numer. Math., 45 (2003), pp. 29–40.
[24] O. L. Mangasarian, *Solution of symmetric linear complementarity problems by iterative methods*, J. Optim. Theory Appl., 22 (1977), pp. 465–485.
[25] I. Marek and D. B. Szyld, *Comparison theorems of weak splittings of bounded operators*, Numer. Math., 58 (1990), pp. 387–397.
[26] K. G. Murty, *Linear Complementarity, Linear and Nonlinear Programming*, Internet Edition 1997.
[27] D. P. O'Leary and R. E. White, *Multisplittings of matrices and parallel solution of linear systems*, SIAM J. Algebraic Discrete Methods, 6 (1985), pp. 630–640.
[28] J.-S. Pang, *Necessary and sufficient conditions for the convergence of iterative methods for the linear complementarity problem*, J. Optim. Theory Appl., 42 (1984), pp. 1–17.
[29] W. M. G. van Bokhoven, *Piecewise Linear Modelling and Analysis*, Proeschrift, Eindhoven, The Netherlands, 1981.
[30] R. S. Varga, *On recurring theorems on diagonal dominance*, Linear Algebra Appl., 13 (1976), pp. 1–9.
[31] R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1962; 2nd revised and expanded ed., Springer, Berlin, 2000.
[32] D. Yuan and Y. Song, *Modified AOR methods for linear complementarity problem*, Appl. Math. Comput., 140 (2003), pp. 53–67.
[33] L.-L. Zhang, *Two-step modulus based matrix splitting iteration method for linear complementarity problems*, Numer. Algorithms, 57 (2011), pp. 83–99.