Journal of Optimization Theory and
Applications, ISSN 0022-3239.
Volume 165, Number 2
J Optim Theory Appl (2015) 165:545-562
DOI 10.1007/s10957-014-0589-4

On the Solution of the Linear Complementarity Problem by the Generalized Accelerated Overrelaxation Iterative Method

Apostolos Hadjidimos · Michael Tzoumas

the date of receipt and acceptance should be inserted later

Abstract In the present work, we determine intervals of convergence for the various parameters involved for what is known as the Generalized Accelerated Overrelaxation iterative method for the solution of the Linear Complementarity Problem. The convergence intervals found constitute sufficient conditions for the Generalized Accelerated Overrelaxation method to converge and are better than what have been known so far.

Keywords linear complementarity problem · H_+-matrices · strictly diagonally dominant (SDD) matrices · projected methods · modulus algorithms · generalized accelerated overrelaxation
AMS Classification Primary 65F10

1 Introduction

The interest in the solution of the Linear Complementarity Problem (LCP) lies in the fact that it has many applications which cover various branches of science, engineering, economics, etc. (see, e.g., [1-3]). Therefore, many direct and iterative methods have been developed for its solution.

One of the oldest iterative methods related to the linear complementarity problem is due to Hildreth [4], who designed the procedure to solve a strictly convex quadratic program. Hildreth stated its Kuhn-Tucker conditions and used the nonsingularity of the Hessian matrix of the objective function to eliminate the primal

Apostolos Hadjidimos (corresponding author)
Department of Electrical & Computer Engineering, University of Thessaly, GR-382 21 Volos, Greece
E-mail: hadjidim@inf.uth.gr

Michael Tzoumas c/o Department of Mathematics, University of Ioannina, GR-451 10 Ioannina, Greece
E-mail: mtzoumas@sch.gr

variables. What remains after this operation is a linear complementarity problem in which the variables are Lagrange multipliers and the coefficient matrix is symmetric and positive semidefinite. A more general iterative method, attributed to Christopherson [5], has been analyzed and clarified by Cryer [6], [7], and it is often cited as Cryer's method. It is a successive overrelaxation (SOR) method proposed for the solution of the free-boundary problem for journal bearings; see also [8], [9]. In the same general direction, excellent works based mainly on the iterative methods for the solution of large sparse linear systems (see, e.g., [10], [11]) were developed by Mangasarian [12], Ahn [13], Pang [14], Pantazopoulos [15] and Koulisianis and Papatheodorou [16]. Meanwhile articles using what was called Modified AOR iterative methods applied to block two-cyclic matrices appeared in [17], [18]. Moreover, three of the most recent works in the class of projected methods are those by Li and Dai [19] and Saberi Najafi and Edalatpanah [20], [21] who further developed what they call "*Generalized AOR*" iterative method; note that on two statements of [19] and Herceg and Cvetković [22], part of the analysis of the method in [20] was based. Finally, a very recent paper, along the lines of the papers in [19-21], first applies preconditioning, and then deals with an application to obstacle problems [23]. (See, also, previous works on the GAOR method cited in the last four articles.)

A quite different class of iterative methods, called "*Modulus Algorithms*", was introduced by van Bokhoven [24] ("*point modulus algorithm*") and was extended by Kappel and Watson [25] to "*block modulus algorithm*". In [26] and [30], "*extrapolation*" was introduced to accelerate the convergence rates of both modulus algorithms. A third class of iterative methods, called "*modulus-based matrix splitting iterative methods*", was introduced by Bai [27] and was extended in many works that followed (see, e.g., [28-34]).

As is known, the LCP possesses a unique solution if and only if the coefficient matrix A is a $P-$matrix; namely, a real matrix whose all principal submatrices have positive determinants (see, e.g, [1-3]). In the present work we consider A to be an H_+-matrix, that is a matrix with positive diagonal and modulus of its associated Jacobi iteration matrix convergent.

The outline of the present work is as follows: In Section 2, a brief description of the main ideas behind the most general case of the Generalized AOR method are sketched. In Section 3, the convergence analysis of two particular GAOR methods follows; these methods depend on two special expressions of a matrix R. In Section 4, the given LCP is transformed into an equivalent one so that the new LCP has its coefficient matrix "*strictly diagonally dominant*" (*SDD*). Also, the new Generalized AOR, which is the union of two distinguished Generalized AORs, is presented. Moreover, sufficient regions of convergence of the parameters involved are found of which the "best" are determined. In Section 5, two numerical examples are presented, a comparison of the methods in [27], [30] and the one in this work is made, and sufficient convergence regions are found. Finally, in Section 6, some remarks conclude our work.

2 Presentation of the GAOR Method

We begin with the formulation of the LCP:
"Let $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. Determine r, $z \in \mathbb{R}^n$, with $r, z \geq 0$, satisfying

$$r = Az + q \quad \text{and} \quad < r, z >= 0." \tag{1}$$

(Note: The vector q has at least one negative component because if $q \geq 0$ then the solution of the LCP is $z = 0$.)

For the solution of the LCP using the GAOR iterative method some basic material is needed.

For any $x \in \mathbb{R}^n$, x_+ is defined to be the vector whose components are nonnegative and are given by $(x_+)_i = \max\{x_i, 0\}\, \forall\, i \in N := \{1, 2, \cdots, n\}$. Then, for any $x, y \in \mathbb{R}^n$ there hold:

$$\begin{array}{ll} i)\ \ (x+y)_+ \leq x_+ + y_+, & ii)\ \ x_+ - y_+ \leq (x-y)_+, \\ iii)\ \ |x| = x_+ + (-x)_+, & iv)\ \ x \leq y \ \Rightarrow\ x_+ \leq y_+. \end{array} \tag{2}$$

(For the properties in (2) see, e.g., [12], [13].)

The main equation of LCP in (1) can be transformed into an equivalent fixed-point equation (see, e.g., [3]) as follows

$$z = \left(z - D^{-1}(Az + q)\right)_+ \quad \Leftrightarrow \quad z = \left(z - \left((I - \widetilde{L} - \widetilde{U})z + \widetilde{q}\right)\right)_+, \tag{3}$$

where A is split in the usual way as $A = D - L - U$, with D, $-L$, $-U$ being the diagonal, the strictly lower and the strictly upper triangular parts of A, respectively, and then we set $\widetilde{L} = D^{-1}L$, $\widetilde{U} = D^{-1}U$, $\widetilde{q} = D^{-1}q$, with I being the identity matrix. If $\widetilde{A} = D^{-1}A$ is split in the form below

$$\widetilde{A} = \Omega^{-1}(I - \alpha\Omega\widetilde{L}) - \Omega^{-1}\left((I - \Omega) + (\Omega - \alpha\Omega)\widetilde{L} + \Omega\widetilde{U}\right), \tag{4}$$

where Ω is a positive (or negative) diagonal matrix and α a real parameter. Then, the original LCP is transformed into the following equivalent forms, where we have set $D^{-1}r = \widetilde{r}$,

$$\begin{array}{l} \Omega\widetilde{r} = \Omega\widetilde{A}z + \Omega\widetilde{q} \ \text{ and } \ (\Omega\widetilde{r})^T z = 0 \ \ \Leftrightarrow \\ \Omega\widetilde{r} = \left((I - \alpha\Omega\widetilde{L}) - \left((I - \Omega) + (\Omega - \alpha\Omega)\widetilde{L} + \Omega\widetilde{U}\right)\right) z + \Omega\widetilde{q} \text{ and } (\Omega\widetilde{r})^T z = 0. \end{array} \tag{5}$$

Note that the conditions in the equivalent relations (5) are obtained from the original LCP by simply multiplying the main equation from the left by ΩD^{-1}. Using the second set of relations of (5) into (3) we obtain that

$$z = \left(z - \left(-\alpha\Omega\widetilde{L}z + \Omega(I - \widetilde{L} - \widetilde{U})z + \alpha\Omega\widetilde{L}z + \Omega\widetilde{q}\right)\right)_+. \tag{6}$$

A careful look into the splitting of $\widetilde{A}$ in (4) reveals that for $\alpha \neq 0$, the splitting in question yields the following iteration matrix associated with $\widetilde{A}$

$$\begin{array}{l} \mathcal{L}_{\alpha\Omega,\Omega} := (I - \alpha\Omega\widetilde{L})^{-1}\left((I - \Omega) + (\Omega - \alpha\Omega)\widetilde{L} + \Omega\widetilde{U}\right) = \\ (I - \alpha\Omega\widetilde{L})^{-1}\left((I - \alpha\Omega\widetilde{L}) - \Omega + \Omega\widetilde{L} + \Omega\widetilde{U}\right) = \\ (1 - \frac{1}{\alpha})I + \frac{1}{\alpha}I + (I - \alpha\Omega\widetilde{L})^{-1}\left(-\Omega + \Omega\widetilde{L} + \Omega\widetilde{U}\right) = \\ (1 - \frac{1}{\alpha})I + \frac{1}{\alpha}(I - \alpha\Omega\widetilde{L})^{-1}\left((I - \alpha\Omega) + \alpha\Omega\widetilde{U}\right) = (1 - \frac{1}{\alpha})I + \frac{1}{\alpha}\mathcal{L}_{\alpha\Omega,\alpha\Omega}. \end{array} \tag{7}$$

So, from the first row of (7) we see that the Generalized AOR is a scalar-matrix-parameter analogue of the scalar-parameter AOR method [35], with $\alpha\Omega =: R$ being the "*acceleration*" and Ω the "*overrelaxation*" matrix-parameters, respectively. On the other hand, from the last row of (7) the most general GAOR is nothing but an extrapolation, with extrapolation parameter $\frac{1}{\alpha}$, of a "Modified SOR", with "*overrelaxation matrix*" $R = \alpha\Omega$. Finally, for $\alpha = 0$, the Generalized AOR is a "*Modified Extrapolated Jacobi*", with "extrapolation matrix" Ω (see, e.g., [36] and some of the references therein).

It becomes clear that by choosing special parameters in the GAOR iterative method we find the other well known iterative methods.

For example:

i) GSOR (Generalized SOR) method for $(R, \Omega) = (\Omega, \Omega)$

ii) AOR (Accelerated Overrelaxation) method for $(R, \Omega) = (rI \neq 0, \omega I)$

iii) MAOR (Modified AOR) method for

$$(R, \Omega) = (\mathrm{diag}(r_1 I_{n_1}, r_2 I_{n_2}), \mathrm{diag}(\omega_1 I_{n_1}, \omega_2 I_{n_2}))$$

for block two-cyclic matrices with $r_1, r_2 \neq 0$ and $n_1 + n_2 = n$, where n_1, n_2 are the orders of the two diagonal blocks

iv) MSOR (Modified SOR) method for

$$(R, \Omega) = (\mathrm{diag}(\omega_1 I_{n_1}, \omega_2 I_{n_2}), \mathrm{diag}(\omega_1 I_{n_1}, \omega_2 I_{n_2}))$$

for block two-cyclic matrices with $n_1 + n_2 = n$, where n_1, n_2 are the orders of the two diagonal blocks

v) SOR method for $(R, \Omega) = (\omega I, \omega I)$

vi) EGS (Extrapolated Gauss-Seidel) method for $(R, \Omega) = (I, \omega I)$

vii) JOR or EJ (Jacobi Overrelaxation or Extrapolated Jacobi) method for $(R, \Omega) = (0, \omega I)$

viii) Gauss-Seidel method for $(R, \Omega) = (I, I)$

ix) Jacobi method for $(R, \Omega) = (0, I)$

In next sections, we will examine the two particular cases of GAOR, where $\alpha = 0$ and $\alpha \neq 0$.

3 Main Theory of the Generalized AOR Method

To study possible convergence of the Generalized AOR (GAOR) method, we begin with equation (6). Then,

$$z^* = \left(z^* - \left(-\alpha\Omega\widetilde{L}z^* + \Omega(I - (1-\alpha)\widetilde{L} - \widetilde{U})z^* + \Omega\widetilde{q}\right)\right)_+, \tag{8}$$

where z^* is the exact solution of our LCP, and with the iterative scheme produced from (6)

$$z^{(k+1)} = \left(z^{(k)} - \left(-\alpha\Omega\widetilde{L}z^{(k+1)} + \Omega(I - (1-\alpha)\widetilde{L} - \widetilde{U})z^{(k)} + \Omega\widetilde{q}\right)\right)_+. \tag{9}$$

Note that an ostensible "*contradiction*" in using the unknown vector $z^{(k+1)}$ in the right side of (9) does not exist because: For $\alpha = 0$, $z^{(k+1)}$ is not present in the

right side. For $\alpha \neq 0$, each component of $z^{(k+1)}$ in the left side depends **only** on previously determined components of $z^{(k+1)}$ of the right side due to the strictly lower triangular nature of $-\alpha\Omega\widetilde{L}$.

A statement similar to the one of Theorem 3.1 in [19] that refers to the connection of two consecutive error vectors of (9) follows. The proof is given analytically for completeness.

Theorem 3.1 Any two consecutive error vectors of iterative scheme (9) are connected via

$$|z^{(k+1)} - z^*| \leq G|z^{(k)} - z^*| \implies \left\|z^{(k+1)} - z^*\right\|_\infty \leq \|G\|_\infty \left\|z^{(k)} - z^*\right\|_\infty, \quad (10)$$

where

$$G \equiv G_{\alpha\Omega,\Omega} := \left(I - \left|\alpha\Omega\right|\left|\widetilde{L}\right|\right)^{-1} \left(\left|I - \Omega\right| + \left|(1-\alpha)\Omega\right|\left|\widetilde{L}\right| + \left|\Omega\right|\left|\widetilde{U}\right|\right) \geq 0. \quad (11)$$

Proof From (8) and (9), using properties (2), we successively obtain [1]

$$\begin{aligned} z^{(k+1)} - z^* &= \left(z^{(k)} - \left(-\alpha\Omega\widetilde{L}z^{(k+1)} + \Omega(I - (1-\alpha)\widetilde{L} - \widetilde{U})z^{(k)} + \Omega\widetilde{q}\right)\right)_+ \\ &- \left(z^* - \left(-\alpha\Omega\widetilde{L}z^* + \Omega(I - (1-\alpha)\widetilde{L} - \widetilde{U})z^* + \Omega\widetilde{q}\right)\right)_+ \overset{(ii)}{\leq} \\ &\left(\left(z^{(k)} - z^*\right) - \left(-\alpha\Omega\widetilde{L}(z^{(k+1)} - z^*) + \Omega(I - (1-\alpha)\ \widetilde{L} - \widetilde{U})(z^{(k)} - z^*)\right)\right)_+. \end{aligned}$$

Hence

$$\begin{aligned} (z^{(k+1)} - z^*)_+ &\leq \left(\alpha\Omega\widetilde{L}(z^{(k+1)} - z^*) + \left(I - \Omega + (1-\alpha)\Omega\widetilde{L} + \Omega\widetilde{U}\right)(z^{(k)} - z^*)\right)_+ \overset{(i)}{\leq} \\ &\left(\alpha\Omega\widetilde{L}(z^{(k+1)} - z^*)\right)_+ + \left(\left(I - \Omega + (1-\alpha)\Omega\widetilde{L} + \Omega\widetilde{U}\right)(z^{(k)} - z^*)\right)_+. \end{aligned} \quad (12)$$

Similarly, we can obtain

$$(z^* - z^{(k+1)})_+ \overset{(i)}{\leq} \left(\alpha\Omega\widetilde{L}(z^* - z^{(k+1)})\right)_+ + \left(\left(I - \Omega + (1-\alpha)\Omega\widetilde{L} + \Omega\widetilde{U}\right)(z^* - z^{(k)})\right)_+. \quad (13)$$

Then, from (12) and (13) we get

$$\begin{aligned} &\left|z^{(k+1)} - z^*\right| \overset{(iii)}{=} (z^{(k+1)} - z^*)_+ + (z^* - z^{(k+1)})_+ \leq \\ &\left(\alpha\Omega\widetilde{L}(z^{(k+1)} - z^*)\right)_+ + \left(\left(I - \Omega + (1-\alpha)\Omega\widetilde{L} + \Omega\widetilde{U}\right)(z^{(k)} - z^*)\right)_+ + \\ &\left(\alpha\Omega\widetilde{L}(z^* - z^{(k+1)})\right)_+ + \left(\left(I - \Omega + (1-\alpha)\Omega\widetilde{L} + \Omega\widetilde{U}\right)(z^* - z^{(k)})\right)_+ \overset{(iii)}{\leq} \\ &\left|\alpha\Omega\widetilde{L}(z^{(k+1)} - z^*)\right| + \left|\left(I - \Omega + (1-\alpha)\Omega\widetilde{L} + \Omega\widetilde{U}\right)(z^{(k)} - z^*)\right| \leq \\ &\left|\alpha\Omega\right|\left|\widetilde{L}\right|\left|z^{(k+1)} - z^*\right| + \left(\left|I - \Omega\right| + \left|(1-\alpha)\Omega\right|\left|\widetilde{L}\right| + \left|\Omega\right|\left|\widetilde{U}\right|\right)\left|z^{(k)} - z^*\right|. \end{aligned} \quad (14)$$

From the extreme members of (14) we take

$$\left(I - \left|\alpha\Omega\right|\left|\widetilde{L}\right|\right)\left|z^{(k+1)} - z^*\right| \leq \left(\left|I - \Omega\right| + \left|(1-\alpha)\Omega\right|\left|\widetilde{L}\right| + \left|\Omega\right|\left|\widetilde{U}\right|\right)\left|z^{(k)} - z^*\right|. \quad (15)$$

[1] A lower case Latin numeral in a pair of parentheses over a relational operator, as, e.g., "$\overset{(ii)}{\leq}$", refers to the application and/or implication of the corresponding property of (2).

Since $\rho\left(\left|\alpha\Omega\right|\left|\widetilde{L}\right|\right) = 0$, the matrix $I - \left|\alpha\Omega\right|\left|\widetilde{L}\right|$ possesses an inverse that has a nonnegative Neumann expansion. Therefore,

$$\left|z^{(k+1)} - z^*\right| \leq \left(I - \left|\alpha\Omega\right|\left|\widetilde{L}\right|\right)^{-1}\left(\left|I - \Omega\right| + \left|(1-\alpha)\Omega\right|\left|\widetilde{L}\right| + \left|\Omega\right|\left|\widetilde{U}\right|\right)\left|z^{(k)} - z^*\right|, \tag{16}$$

and the first relation in (10) is obtained. If G is the matrix in (11), and (16), then taking $\ell_\infty-$norms the second relation in (10) is also obtained. □

Remark 3.1 It is noted that $G \geq \left|\mathcal{L}_{\alpha\Omega,\Omega}\right|$, where $\mathcal{L}_{\alpha\Omega,\Omega}$ is the Generalized AOR iteration matrix in (7). So, $G = G_{\alpha\Omega,\Omega}$ is a "*majorizer*" of $\left|\mathcal{L}_{\alpha\Omega,\Omega}\right|$ and it is G that will play the key-role in our analysis. This is because for either scheme in (10) to converge, the error of its iterates must have an R-convergence factor, in the sense of Ortega and Rheinboldt [37]. Its supremum for all initial errors is the convergence factor of the iterative method. This supremum is given by $\rho(G)$. A sufficient condition for convergence is then $\rho(G) < 1$.

Remark 3.2 There are α's and Ω's for which either scheme mentioned in Remark 3.1 converges can be proven very easily. For $\alpha = 0$ and $\Omega = I$, G becomes the modulus of the Jacobi iteration matrix which converges since A and $\widetilde{A}$ is an H_+-matrix. Because the eigenvalues of G, and $\rho(G)$, are continuous functions of the elements of G, there will exist $\epsilon > 0$ sufficiently small, such that $\rho(G) < 1$ for all $\alpha \in \,]-\epsilon, +\epsilon\,[$ and all Ω, with $(1-\epsilon)I \leq \Omega \leq (1+\epsilon)I$.

It remains to determine the values of α and of the diagonal elements of the matrix Ω ($\omega_i \;\forall\, i \in N$) to make $\|G\|_\infty$ be strictly less that 1. This is done in the next section.

4 Determination of the Parameters Involved

4.1 Transformation of the Original LCP

To analyze the convergence of the Generalized AOR iterative method, we have to transform our problem so that the coefficient matrix of the LCP is an SDD one. Since A, or

$$\widetilde{A} = D^{-1}A = D^{-1}(D - L - U) = I - \widetilde{L} - \widetilde{U},$$

is an H_+-matrix it suffices to find a positive diagonal matrix E so that $\widetilde{A}E$ or $E^{-1}\widetilde{A}E$ is SDD. There are many algorithms to determine such an E matrix. One of them is a slight modification of the Algorithm in [38] (or [39]) which determines such a matrix E or a "good" approximation from above to the upper bound below

$$\max_{i=1(1)n}\left(\frac{1}{e_{ii}}\sum_{j=1,\; j\neq i}^{n}(|\widetilde{a}_{ij}|e_{jj})\right)\;(<1)\,.$$

Having determined E, the LCP in (5) becomes

$$\begin{array}{l} E^{-1}\Omega\widetilde{r} = E^{-1}\Omega\widetilde{A}EE^{-1}z + E^{-1}\Omega\widetilde{q} \;\text{ and }\; (E^{-1}\Omega\widetilde{r})^T(E^{-1}z) = 0 \;\;\Leftrightarrow \\ \Omega\widehat{r} = \Omega\widehat{A}\widehat{z} + \Omega\widehat{q} \;\text{ and }\; (\Omega\widehat{r})^T\widehat{z} = 0, \end{array} \tag{17}$$

where we have set

$$\widehat{r} = E^{-1}\widetilde{r},\;\; \widehat{A} = E^{-1}\widetilde{A}E,\;\; \widehat{z} = E^{-1}z,\;\; \widehat{q} = E^{-1}\widetilde{q}. \tag{18}$$

In this way, we end up with a new LCP, in (17), equivalent to the original one, whose matrix coefficient $\Omega\widehat{A}$ (and $\widehat{A}$) is an SDD matrix and, in addition, $\widehat{A}$ is similar to $\widetilde{A}$.

Remark 4.1 Obviously, the new LCP (17) has a unique solution. Therefore, after $\widehat{r}$ and $\widehat{z}$ are determined the vectors r and z of the original LCP (1) are recovered by multiplying the determined vectors, the former by ED and the latter by E.

4.2 **Generalized AOR**

In next two subsections, we examine first the Generalized AOR with $\alpha = 0$ and then the Generalized AOR with $\alpha \neq 0$. Note that the GAOR with $\alpha = 0$ admits a full exploitation of its inherent parallelism and thus it may be faster than the "best" GAOR with $\alpha \neq 0$. The "best" of the latter method is determined in a third subsection.

4.2.1 **Generalized AOR with** $\alpha = 0$

In this case it is

$$\widehat{G} \equiv \widehat{G}_{0,\Omega} = \big|I - \Omega\big| + \big|\Omega\big|\big|\widehat{L}\big| + |\Omega|\big|\widehat{U}\big| \geq 0.$$

Clearly, $\|\widehat{G}\|_\infty < 1$ if and only if

$$|1-\omega_i|+|\omega_i|(l_i+u_i) < 1 \ \forall\, i \in N, \text{ where } l_i = \sum_{j=1}^{i-1} |l_{ij}|,\ l_1 = 0,\ u_i = \sum_{j=i+1}^{n} |u_{ij}|,\ u_n = 0, \tag{19}$$

with l_{ij} and u_{ij} being the elements of the lower and upper triangular parts of $\widehat{L}$ and $\widehat{U}$, respectively. Recalling that $l_i + u_i =: \eta_i < 1\ \forall\, i \in N$, the ω_i's can not be negative since then $|1 - \omega_i| > 1 \forall\, i \in N$ are not making inequalities (19) possible. Hence, $\omega_i > 0\ \forall\, i \in N$. From (19) we readily obtain

$$\omega_i \in \,]\,0,\ \frac{2}{1+\eta_i}\,[\ \ \forall\, i \in N. \tag{20}$$

For the "best" GAOR it is proved that the function $|1 - \omega_i| + \omega_i\eta_i$ is strictly decreasing in the interval $\omega_i \in \,]\,0,\,1\,]$ and strictly increasing in $\omega_i \in [\,1,\ \frac{2}{1+\eta_i}\,[$. Consequently, the "best" GAOR is that for $\omega_i = 1\ \forall\, i \in N$. It is then the Jacobi method itself whose $\ell_\infty-$norm of the associated iteration matrix is

$$\big\|\mathcal{L}_{0,I}\big\|_\infty \leq \big\|\widehat{G}_{0,I}\big\|_\infty = \max_{i=1(1)n} \eta_i =: \eta. \tag{21}$$

4.2.2 **Generalized AOR with** $\alpha \neq 0$

We begin with relation (11), as it has already been transformed with the hatted entities, from which we have that

$$(I - |\alpha|\big|\Omega\big|\big|\widehat{L}\big|)\widehat{G} = \big|I - \Omega\big| + |1 - \alpha|\big|\Omega\big|\big|\widehat{L}| + \big|\Omega\big|\big|\widehat{U}|. \tag{22}$$

Equating the (i,j)th elements of the matrices in the two sides of the equation in (22) we have

$$\widehat{g}_{ij} - |\alpha||\omega_i| \sum_{k=1}^{i-1} (|l_{ik}|\widehat{g}_{kj}) = (|I - \Omega|)_{ij} + |1-\alpha||\omega_i||l_{ij}| + |\omega_i||u_{ij}|,$$

where l_{ij} and u_{ij} are the elements of $\widehat{L}$ and $\widetilde{U}$ as in the previous case. Note that the left side above comes from the product of the ith row of $(I - |\alpha|\big|\Omega\big|\big|\widehat{L}\big|)$ and the jth column of $\widehat{G}$ and the right side from the (i,j)th elements of the three matrices involved. Then, solving for the $\widehat{g}_{ij}$ element of the matrix $\widehat{G}$ we get

$$\widehat{g}_{ij} = |\alpha||\omega_i| \sum_{k=1}^{i-1} (|l_{ik}|\widehat{g}_{kj}) + |1-\alpha||\omega_i||l_{ij}| + (|I - \Omega|)_{ij} + |\omega_i||u_{ij}| \quad \forall\, i \in N. \tag{23}$$

Denoting by $\widehat{g}_i$, $i \in N$, the ith row sum of the nonnegative elements of $\widehat{G}$, we successively obtain

$$\begin{aligned} \widehat{g}_i = \textstyle\sum_{j=1}^{n} \widehat{g}_{ij} &= |\alpha||\omega_i| \textstyle\sum_{j=1}^{n} \left(\sum_{k=1}^{i-1} (|l_{ik}|\widehat{g}_{kj}) \right) + |1-\alpha||\omega_i|l_i + |1-\omega_i| + |\omega_i|u_i \\ &= |\alpha||\omega_i| \textstyle\sum_{k=1}^{i-1} \left(|l_{ik}| \sum_{j=1}^{n} (\widehat{g}_{kj}) \right) + |1-\alpha||\omega_i|l_i + |1-\omega_i| + |\omega_i|u_i \\ &= |\alpha||\omega_i| \textstyle\sum_{k=1}^{i-1} (|l_{ik}|\widehat{g}_k) + |1-\alpha||\omega_i|l_i + |1-\omega_i| + |\omega_i|u_i \quad (\forall\, i \in N), \end{aligned} \tag{24}$$

where $l_i, u_i \,\forall\, i \in N$, were introduced in (19). Clearly,

$$||\widehat{G}||_\infty = \max_{i=1(1)n} \widehat{g}_i < 1 \;\text{ or }\; \widehat{g}_i < 1 \;\forall\, i \in N$$

hold if and only if the corresponding parameter ω_i in the right side of the expression for $\widehat{g}_i$ in (24) is positive. For if ω_i were negative then the term $|1-\omega_i|$ would be greater than 1 and the corresponding $\widehat{g}_i$ would also be greater than 1. Therefore, (24) becomes

$$\widehat{g}_i = |\alpha|\omega_i \sum_{k=1}^{i-1} (|l_{ik}|\widehat{g}_k) + |1-\alpha|\omega_i l_i + |1-\omega_i| + \omega_i u_i \quad (\forall\, i \in N). \tag{25}$$

From the right expression above, each $\widehat{g}_i$ is a complicated polynomial expression of degree $i-1$ in terms of α. This makes the minimization of $\widehat{g}_i$, at least for $i > 2$, a very difficult problem to handle even for relatively small values of n. So, in the cases which are to be examined we will use only upper bounds for the $\widehat{g}_k$'s, $k = 1(1)i-1$, to find an upper bound for $\widehat{g}_i \,\forall\, i \in N$, especially in Section 4.3. Note that we will keep the same notation $\widehat{g}_i$ for the new bounds obtained from (25).

We begin by distinguishing cases depending on whether $\omega_i \in \,]\,0,\ 1]$ or $\omega_i \geq 1$.

$\omega_i \in \,]\,0,\ 1]$: Observe that for $i = 1$, $\widehat{g}_1 = 1 - \omega_1(1-u_1) < 1$ since $u_1 \in [0,1[$. Using induction we assume that $\widehat{g}_k < 1 \;\forall\, k = 1(1)i-1, i \in N\backslash\{1\}$, and will prove that $\widehat{g}_i < 1$. From (25), to have $\widehat{g}_i < 1$ it is equivalent to have the condition below

$$|\alpha| \sum_{k=1}^{i-1} (|l_{ik}|\widehat{g}_k) + |1-\alpha|l_i + u_i < 1, \tag{26}$$

be satisfied. By the induction hypothesis, $\widehat{g}_k < 1$, $k = 1(1)i-1$, (26) will hold if the following sufficient condition holds

$$(|\alpha| + |1-\alpha|)\, l_i + u_i < 1, \ \forall\, i \in N\backslash\{1\}. \tag{27}$$

Denoting by

$$m := -\min_{i=1(1)n,\, l_i \neq 0} \left\{ \frac{1-\eta_i}{2l_i} \right\} \quad \text{and} \quad M := \min_{i=1(1)n,\, l_i \neq 0} \left\{ \frac{1-u_i+l_i}{2l_i} \right\} \tag{28}$$

the two quantities that will frequently appear in the sequel, we distinguish cases in order to get rid of the moduli involved.

$\alpha \in]-\infty, 0[$: In this case we have $(1-2\alpha)l_i + u_i < 1$ or $-2\alpha l_i < 1 - l_i - u_i$ which gives $-\alpha < \min_{i=1(1)n} \left\{ \frac{1-\eta_i}{2l_i} \right\}$ or, equivalently, $\alpha \in \,]\, m,\, 0\,[$. (Note: In the present case and in the ones that will follow if any $l_i = 0$, then the first inequality, $(1-2\alpha)l_i + u_i < 1$, becomes $u_i < 1$ and so for the corresponding inequality there holds $\alpha \in]-\infty, 0[$. Hence, the minimum is taken over all l_i's > 0.)

$\alpha \in [0, 1]$: From $(\alpha + 1 - \alpha)l_i + u_i < 1$ or $\eta_i < 1$, which always holds.

$\alpha \in]1, +\infty[$: This time $(2\alpha - 1)l_i + u_i < 1$ is equivalent to $\alpha \in \,]\, 1,\, M\,[$. (Note: Again, if $l_i = 0$ then the corresponding $\alpha \in]1, +\infty[$.)

Collecting all the results from all three cases for α, we obtain

Theorem 4.1 Under the assumption $\omega_i \in]0, 1]$, the notation so far and the preceding analysis, the conditions below

$$\omega_i \in \,]\, 0, 1]\ \text{ and } \ \alpha \in \,]\, m,\, M\,[\tag{29}$$

constitute sufficient ones for $\left\| \mathcal{L}_{\alpha\Omega,\Omega} \right\|_\infty \leq \left\| \widehat{G}_{\alpha\Omega,\Omega} \right\|_\infty \leq \max_{i\in N} \widehat{g}_i < 1$.

$\omega_i \in [1, +\infty[$: As before, in order to have $\widehat{g}_i < 1\, \forall\, i \in N$, use of induction is made. So, for $i = 1$ we must have from (25) that $\omega_1 \in [1\ ,\ \frac{2}{1+u_1}\,[$. Under the assumption $\widehat{g}_k < 1$, $k = 1(1)i-1$, from (25) it is seen that it suffices to have

$$\omega_i \in [\ 1,\ \frac{2}{1 + (|a| + |1-a|)l_i + u_i}\,[\quad \text{or} \quad (|a| + |1-a|)l_i + u_i < 1,\ i \in N\backslash\{1\}, \tag{30}$$

which is identical to that in (27). The three cases for α considered before are considered again and the end result is a proposition similar to Theorem 4.1 which is given below.

Theorem 4.2 Under the assumption $\omega_i \geq 1$, the notation so far and the preceding analysis, the conditions below

$$\omega_i \in [\ 1,\ \frac{2}{1 + (|a| + |1-a|)l_i + u_i}\,[\quad \text{and} \quad \alpha \in \,]\, m,\, M\,[\tag{31}$$

constitute sufficient ones for $\left\| \mathcal{L}_{\alpha\Omega,\Omega} \right\|_\infty \leq \left\| \widehat{G}_{\alpha\Omega,\Omega} \right\|_\infty \leq \max_{i\in N} \widehat{g}_i < 1$.

Remark 4.2 The results of Theorems 4.1 and 4.2 are more accurate than the corresponding ones in [20] due to the more accurate expressions for the $\widehat{g}_i$'s in (24). Observe that, in the aforementioned theorems, the intervals for α are identical.

4.3 "Best" Parameters of Generalized AOR

With fixed $\omega_i\ \forall\, i \in N$ in their convergence intervals, our effort will be to minimize the expression for $\widehat{g}_i$, $i = 2(1)n$, in (25), assuming that $\widehat{g}_k < 1$ for all $k = 1(1)i-1$. It is

$$\widehat{g}_1 = 1 - \omega_1(1-u_1) < 1 \ \text{ for } \ \omega_1 \in \,]\, 0, 1\,]$$

and

$$\widehat{g}_1 = \omega_1(1+u_1) - 1(<1) \quad \text{for} \quad \omega_1 \in [\,1, \frac{2}{1+u_1}\,[\,.$$

As in the previous section we distinguish three cases depending on the ordering of α, 0 and 1.

$\alpha \in]\,m,\,0\,]$: Differentiating (25) with respect to α we obtain

$$\frac{\partial \widehat{g}_i}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(|\alpha|\omega_i \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) + |1-\alpha|\omega_i l_i + |1-\omega_i| + \omega_i u_i\right) = \\ -\omega_i \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) - \omega_i l_i < 0.$$

So, $\widehat{g}_i$ is strictly decreasing in $]\,m,\,0]$ and its minimum value is assumed at $\alpha = 0$.

$\alpha \in [0,1]$: This time we have

$$\frac{\partial \widehat{g}_i}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(|\alpha|\omega_i \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) + |1-\alpha|\omega_i l_i + |1-\omega_i| + \omega_i u_i\right) = \\ \omega_i \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) - \omega_i l_i < 0,$$

because $\widehat{g}_k < 1$, $k = 1(1)i-1$. Therefore, the function of interest continues to be strictly decreasing in $[0,1]$ and its minimum is attained at $\alpha = 1$.

$\alpha \in [1\ ,\ M\,[$: It is

$$\frac{\partial \widehat{g}_i}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(|\alpha|\omega_i \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) + |1-\alpha|\omega_i l_i + |1-\omega_i| + \omega_i u_i\right) = \\ \omega_i \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) + \omega_i l_i > 0.$$

The function in question is strictly increasing. Hence its minimum is attained at $\alpha = 1$.

As a conclusion from all three cases examined we have that the minimum of $\widehat{g}_i\ \forall\, i \in N\backslash\{1\}$ is assumed at $\alpha = 1$ and then, we have

$$\min_{\alpha \in\,]m,\, M[}\ \widehat{g}_i(\alpha) = \widehat{g}_i(1) = \omega_i \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) + |1-\omega_i| + \omega_i u_i,\ \ i \in N. \tag{32}$$

Therefore, we have just proved that

Theorem 4.3 For any fixed $\omega_i > 0$, the smallest value for each $\widehat{g}_i\ \forall\, i \in N\backslash\{1\}$, as a function of α, is attained at $\alpha = 1$.

To find the best value of $\omega_i\ \forall\, i \in N$, we have to distinguish the two cases $\omega_i \in]\,0,1\,]$ and $\omega_i \in [\,1, \frac{2}{1+\eta_i}\,[$. It is readily found out that the function

$$\widehat{g}_i(1) = \omega_i \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) + |1-\omega_i| + \omega_i u_i,\ \ i \in N, \tag{33}$$

is strictly decreasing with ω_i increasing in $]\,0,1]$ and strictly increasing with ω_i in $[\,1, \frac{2}{1+\eta_i}\,[\,.$

As a result we have that

Theorem 4.4 The "best" Generalized AOR iterative method has parameters $\alpha = 1$ and $\Omega = I$, namely it is nothing but the Gauss-Seidel iterative method whose "best" $\ell_\infty-$norm of its iteration matrix is given by

$$||\mathcal{L}_{I,I}||_\infty \le ||\widehat{G}_{I,I}||_\infty = \max\left\{\widehat{g}_1 = u_1,\ \widehat{g}_i = \sum_{k=1}^{i-1}(l_{ik}\widehat{g}_k) + u_i\ \forall\, i \in N\backslash\{1\}\right\} < 1. \tag{34}$$

5 Numerical Examples

The numerical examples given below compare the "best" Generalized AOR method of this work, namely the Generalized GS with $\alpha = 1$, against the "best" Modulus AOR method of [30], that is the Modulus GS one, and Modulus GS method of [27]

Example 5.1 The main difference in the theory of Modulus-based matrix splitting AOR (MAOR) methods of [27] and [30] lies in the fact that in the first method the matrix $\Omega = \frac{1}{2}\text{diag}(A)$ while in the second one is $\Omega = \text{diag}(A)$. The "best" method found in [30] was the Modulus-based matrix splitting Gauss-Seidel (MGS) method. It should be worth mentioning that the methods in the works that followed [30], that is [31], [33] and [34], the former two considered as single-splitting methods, are based on the same splitting as the one in [30]. Their "best" Modulus AOR methods is the Modulus GS method as was proven in [34]; that is the same as that in [30]. To compare the methods in [27] and [30] with the method of this paper, with $\alpha = 1$ and $\Omega = I$, we use Examples 5.1 and 5.2 of [27]. In our computations $z^{(0)} = [0, 0, 0, 0, ...]^T \in \mathbb{R}^n$, $n = m^2$. Our runs were performed using MATLAB 7.9. For each case five runs were performed the extreme two values for CPU times were discarded and it is the average of the other three that is presented in Tables 1 and 2. The stopping criterion imposed in all the runs of both examples was $||\bar{z} - z^*||_\infty \leq 0.5 \times 10^{-14}$, with $\bar{z}$ being the computed solution and z^* the exact one. Note that the vectors z^* and r^* were given for each problem, while q was taken to be $q = r^* - Az^*$.

Example 5.1.1 (Example 5.1 of [27]). Let LCP (1.1) with

$$\begin{gathered} A = \bar{A} + 4I_n \in \mathbb{R}^{n\times n}, \bar{A} = \text{tridiag}(-I_m, S, -I_m) \in \mathbb{R}^{n\times n}, \\ S = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{m\times m}, \\ z^* = [1, 2, 1, 2, ...]^T \in \mathbb{R}^n, r^* = [0, 0, 0, 0, ...]^T \in \mathbb{R}^n. \end{gathered} \tag{35}$$

The results obtained are illustrated in Table 1. From these results we can observe that for rather small values of n the best of the three methods is the Modulus GS one of [30] while as n increases the "best" Generalized GS with $\alpha = 1$ is the best of the three methods. The latter two methods outperform the Modulus GS one of [27].

Table 1 Example 5.1.1

	Modulus GS (of [27])		Modulus GS (of [30])		Generalized GS (of this work ($\alpha = 1$))	
n	iter	CPU	iter	CPU	iter	CPU
100	78	0.083	38	0.052	28	0.031
400	86	0.427	40	0.271	31	0.291
900	86	1.745	40	0.750	31	0.953
1600	86	5.338	40	2.302	31	2.969
2500	86	12.967	40	5.578	31	6.557

Example 5.1.2 (Example 5.2 of [27]). This time the differences from the previous Example 5.1.1 are that

$$\bar{A} = \text{tridiag}(-1.5I_m, S, -0.5I_m) \in \mathbb{R}^{n\times n} \text{ and } S = \text{tridiag}(-1.5, 4, -0.5) \in \mathbb{R}^{m\times m}. \tag{36}$$

From Table 2, the conclusion that can be drawn as regards to the performance of the three methods examined previously is that the "best" Modulus GS of [30] and the "best" Generalized GS method of this work are more or less competitive and both outperform the corresponding one in [27].

Table 2 Example 5.1.2

	Modulus GS (of [27])		Modulus GS (of [30])		Generalized GS (of this work ($\alpha = 1$))	
n	iter	CPU	iter	CPU	iter	CPU
100	55	0.063	33	0.048	21	0.021
400	56	0.354	35	0.130	21	0.177
900	56	1.245	35	0.656	21	0.693
1600	56	3.625	35	2.031	21	1.906
2500	56	8.635	35	4.969	21	4.500

The following small example uses the matrix M, our A, of Example 3.1 of [20].
Example 5.2*:* The coefficient matrix A in (1) is given by

$$A = \begin{bmatrix} 1 & 0.1 & 0.3 \\ 0.2 & 1 & 0.4 \\ 0.5 & 0.3 & 1 \end{bmatrix}. \tag{37}$$

As is seen A is a SDD matrix, and so an H_+-matrix, with unit diagonal elements. This simply means that $\widehat{A} = \widetilde{A} = A$ and, therefore, all the entities associated with $\widehat{A}$, and $\widetilde{A}$, are identical to those associated with A.
a) For the convergence intervals of the GAOR iteration matrix, with $\alpha = 0$, we have $\Omega = \text{diag}(\omega_1, \omega_2, \omega_3)$ and, from (20),

$$\omega_1 \in \,]\,0, \frac{10}{7}\,[\quad \omega_2 \in \,]\,0, \frac{5}{4}\,[\,,\quad \omega_3 = \,]\,0, \frac{10}{9}\,[\,, \tag{38}$$

while the "best" GAOR of this class is that for $\Omega = I$.
b) For the convergence intervals of the GAOR, with $\alpha \neq 0$, we have: From Theorems 4.1 and 4.2, which give the same interval for α, we can find that the intersection of the three intervals is $\alpha \in \,]\,-0.125, 1.125\,[$. For the corresponding ω_i's we have

$$\omega_1 \in \,]\,0, \tfrac{10}{7}\,[\,,\quad \omega_2 \in \,]\,0, \tfrac{2}{1+0.2(|\alpha|+|1-\alpha|)+0.4}\,[\,,\quad \omega_3 \in \,]\,0, \tfrac{2}{1+0.8(|\alpha|+|1-\alpha|)}\,[\,. \tag{39}$$

Note that for $\alpha = 0$, the intervals in (39) become those in (38). For $\omega_1, \omega_2, \omega_3$ as functions of $\alpha \in \,]\,-0.125, 1.125\,[$, we have from (39) that

$$\begin{array}{rcl} \text{for } \alpha \in \,]\,-0.125, 0] & \Rightarrow & \omega_1 \in \,]\,0, \frac{10}{7}\,[\,,\ \omega_2 \in \,]\,0, \frac{5}{4-\alpha}\,[\,,\ \omega_3 \in \,]\,0, \frac{10}{9-10\alpha}\,[\,, \\ \text{for } \alpha \in [0, 1] & \Rightarrow & \omega_1 \in \,]\,0, \frac{10}{7}\,[\,,\ \omega_2 \in \,]\,0, \frac{5}{4}\,[\,,\ \omega_3 \in \,]\,0, \frac{10}{9}\,[\,, \\ \text{for } \alpha \in [1, 1.125\,[& \Rightarrow & \omega_1 \in \,]\,0, \frac{10}{7}\,[\,,\ \omega_2 \in \,]\,0, \frac{5}{3+\alpha}\,[\,,\ \omega_3 \in \,]\,0, \frac{10}{1+8\alpha}\,[\,, \end{array} \tag{40}$$

while for the "best" GAOR of this class we have $\alpha = 1$ and $\Omega = I$.

Figure 1 illustrates the regions of convergence for $\alpha \in \,]-0.125, 1.125[$ and for $\omega_1, \omega_2, \omega_3$ given from (40). The upper bound for ω_1 is a straight line while the upper bounds for ω_2 and ω_3 consist of three segments; those on the left and on the right are pieces of the corresponding hyperbolas given in (40) while the ones in the middle are straight line segments.

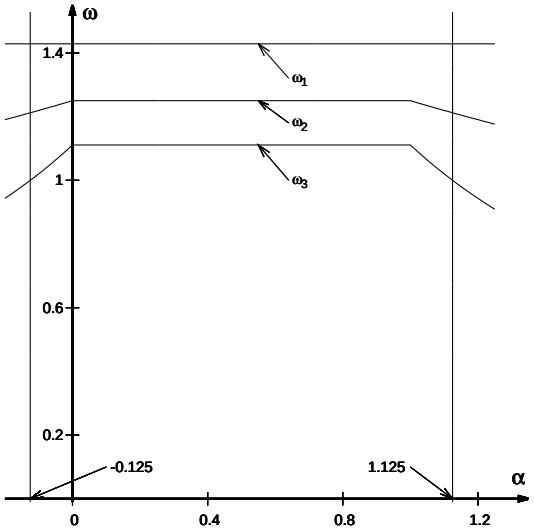

Fig. 1 Region of convergence of Example 5.2 with $\omega_1, \omega_2, \omega_3$ given from (39)

In Table 3 the spectral radius of the matrix G is given for $\alpha = -0.1(0.1)1.1$ (horizontal axis) and for $\omega = \omega_1 = \omega_2 = \omega_3 \in \,]0.1, 1.254[$ (vertical axis). As is seen the increment in the ω_i's was 0.1 in the beginning and towards the end it became finer up to where $\rho(G) \approx 1$. As can be verified, for each α in the table the smallest $\rho(G(\alpha))$ is assumed at $\omega = 1$ and the minimum $\rho(G(\alpha, \omega))$ takes place at $(\alpha, \omega) = (1, 1)$ as our theory predicts.

Figure 2 depicts the sufficient convergence region Table 3 presents.

6 Conclusions

In the present work, we dealt with the GAOR method that constitutes the scalar-matrix-parameter analogue of the scalar-parameter method which is nothing but the classical AOR method (see [35]).

In Section 3, and especially in Theorem 3.1, we proved the convergence of the above method in the cases $\alpha \neq 0$ and $\alpha = 0$. Then, we are able to compare our work with similar ones that have appeared in the very recent works [19]-[21].

In Section 4, we found sufficient conditions for the intervals of the real parameter α and of the diagonal elements of the (positive) diagonal matrix Ω so that the GAOR method converges. We worked with the $\ell_\infty-$norm of the majorizer matrix G in (11) and **not** with its spectral radius as this was done in [19]; this we intend

Table 3 Spectral radius of matrix G, $\rho(G)$

	-0.100	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000	1.100
0.100	0.9651	0.9595	0.9594	0.9592	0.9591	0.9590	0.9589	0.9588	0.9586	0.9585	0.9584	0.9582	0.9641
0.200	0.9299	0.9190	0.9185	0.9180	0.9175	0.9170	0.9165	0.9160	0.9154	0.9149	0.9143	0.9138	0.9258
0.300	0.8946	0.8785	0.8774	0.8763	0.8751	0.8739	0.8727	0.8715	0.8702	0.8689	0.8676	0.8662	0.8849
0.400	0.8591	0.8380	0.8360	0.8340	0.8319	0.8298	0.8275	0.8252	0.8228	0.8203	0.8177	0.8150	0.8409
0.500	0.8234	0.7974	0.7944	0.7912	0.7879	0.7844	0.7808	0.7770	0.7730	0.7688	0.7643	0.7596	0.7934
0.600	0.7875	0.7569	0.7525	0.7479	0.7430	0.7378	0.7324	0.7266	0.7204	0.7138	0.7067	0.6991	0.7417
0.700	0.7514	0.7164	0.7104	0.7040	0.6972	0.6899	0.6822	0.6738	0.6648	0.6549	0.6441	0.6321	0.6849
0.800	0.7151	0.6759	0.6680	0.6595	0.6504	0.6406	0.6300	0.6184	0.6056	0.5914	0.5754	0.5568	0.6219
0.900	0.6785	0.6354	0.6253	0.6145	0.6027	0.5899	0.5758	0.5601	0.5425	0.5223	0.4986	0.4697	0.5507
1.000	0.6418	0.5949	0.5824	0.5688	0.5539	0.5375	0.5192	0.4984	0.4745	0.4461	0.4108	0.3631	0.4681
1.100	0.8111	0.7544	0.7460	0.7370	0.7272	0.7164	0.7045	0.6912	0.6761	0.6590	0.6390	0.6149	0.7172
1.200	0.9813	0.9139	0.9107	0.9072	0.9035	0.8994	0.8950	0.8900	0.8846	0.8785	0.8716	0.8638	0.9717
1.210	0.9984	0.9298	0.9272	0.9244	0.9213	0.9180	0.9143	0.9103	0.9058	0.9008	0.8952	0.8889	0.9976
1.2109	0.9999	0.9312	0.9287	0.9259	0.9229	0.9196	0.9160	0.9121	0.9077	0.9028	0.8974	0.8912	0.9999
1.211	−−	0.9314	0.9288	0.9261	0.9231	0.9198	0.9162	0.9123	0.9079	0.9031	0.8976	0.8914	−−
1.220	−−	0.9458	0.9437	0.9415	0.9391	0.9365	0.9337	0.9305	0.9271	0.9232	0.9189	0.9141	−−
1.230	−−	0.9617	0.9602	0.9587	0.9570	0.9551	0.9531	0.9509	0.9484	0.9457	0.9427	0.9393	−−
1.240	−−	0.9776	0.9768	0.9759	0.9749	0.9738	0.9726	0.9713	0.9699	0.9683	0.9665	0.9645	−−
1.250	−−	0.9936	0.9934	0.9931	0.9928	0.9925	0.9921	0.9918	0.9914	0.9909	0.9904	0.9898	−−
1.251	−−	0.9952	0.9950	0.9948	0.9946	0.9944	0.9941	0.9938	0.9935	0.9932	0.9928	0.9924	−−
1.252	−−	0.9968	0.9967	0.9965	0.9964	0.9962	0.9961	0.9959	0.9957	0.9954	0.9952	0.9949	−−
1.253	−−	0.9984	0.9983	0.9983	0.9982	0.9981	0.9980	0.9979	0.9978	0.9977	0.9976	0.9974	−−
1.254	−−	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	−−

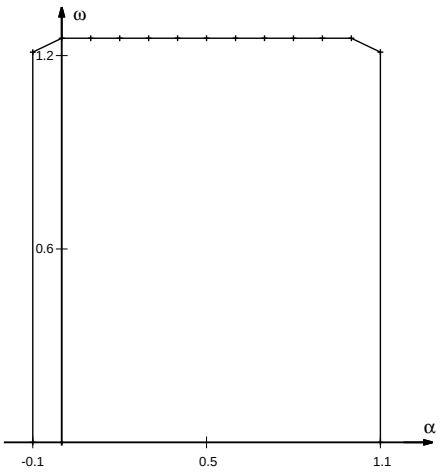

Fig. 2 Region of convergence of Example 5.2 with $\Omega = I$

to do in a future work. We also found the "best" values for the aforementioned parameters and not the "optimal" ones because use of the $\ell_\infty-$norm of the majorizer matrix G was made and this norm is a very complicated polynomial function of α.

For the sufficient regions of convergence we found we may comment as follows: a) In [19], the parameter α was restricted to the interval $[0, 1]$. In our work the interval for α was extended below 0 and above 1 b) In [20], cases of α below 0 and above 1 were also considered. However, the formulas for the convergence intervals in Theorems 4.1 and 4.2 are better than the corresponding ones in Theorem 3.1 of [20].

As regards the numerical examples presented in Section 5, Examples 5.1.1 and 5.1.2, the method in [30] is better than the other two in the former example and comparable to the one in this work in the latter example. Both methods outperform the corresponding method in [27]. However, it should be said that much experimental work has to be done before one decides which of the two

methods should be used in a certain real application. We hope that this work will help researchers to work in this direction.

Example 5.2 is a simple one taken from [20]. In Figure 1 the sufficient region of convergence is illustrated for the different values of ω_i, $i = 1, 2, 3$, as this was found in Section 5. Table 3 gives the aforementioned sufficient convergence region for $\alpha = -0.1(0.1)1.1$ and $\omega = \omega_1 = \omega_2 = \omega_3$. As is readily seen the smallest $\rho(G)$ is attained at $\alpha = 1$ and $\Omega = I$. Also, as ω was increasing above 1.2 the increment became finer in order to find out up to which value of ω we had convergence for each of the selected values for α. For illustrative purposes the above convergence region, based on Table 3, is depicted in Figure 2.

Before we conclude we would like to say that the analysis in the present work followed, in general, previous ones but in more depth and that is why our results seem to be better than those in similar works and comparable to the ones in [30]. However, we did not try to exploit the theory of regular splittings (see Varga [10]), as this was done partly in [30] and entirely in [34]. We have been working in this direction and if we are successful we may come up with even better results.

Acknowledgments The authors are most grateful to the anonymous referees for their very detailed and constructive criticism on a previous version of this work which greatly improved the quality of the present paper.

References

1. Berman A., Plemmons R. J.: Nonnegative Matrices in the Mathematical Sciences. Classics in Applied Mathematics. SIAM, Philadelphia (1994)
2. Cottle R. W., Pang J.-S., Stone R. E.: The Linear Complementarity Problem. Computer Science and Scientific Computing. Academic Press, Boston (1992)
3. Murty K. G.: Linear Complementarity, Linear and Nolinear Programming. Internet Edition (1997)
4. Hildreth C.: Point estimates of ordinates of concave function. J. American Stat. Assoc. 49, 598–619 (1954)
5. Christopherson D. G.: A new mathematical method for the solution of film lubrication problems. Institute of Mechanical Engineers, Proceedings 146, 126–135 (1941)
6. Cryer C. W.: The method of Christopherson for solving free boundary problems for infinite journal bearings by means of finite differences. Math. Comput. 25, 435–443 (1971)
7. Cryer C. W.: The solution of a quadratic programming problem using systematic overrelaxation. SIAM J. Control 9, 385–392 (1971)
8. Rainondi A. A., Boyd J.: A solution for the finite journal bearing and its application to analysis and design, III. Trans. American Soc. of Lubric. Engineers 1, 194–209 (1958)
9. Fridman V. M., Chernina V. S.: An iteration process for the solution of the finite dimensional contact problem. USSR Comput. Math. Math. Physics 8, 210–214 (1967)
10. Varga R. S.: Matrix Iterative Analysis. 2nd Edition, Revised and Expanded, Springer, Berlin (2000)
11. Young D. M.: Iterative Solution of Large Linear Systems. Academic Press, New York (1971)
12. Mangasarian O. L.: Solution of symmetric linear complementarity problems by iterative methods. J. Opt. Theory Appl. 22, 465–485, (1977)
13. Ahn B. H.: Solution of nonsymmetric linear complementarity problems by iterative methods. J. Opt. Theory Appl. 33, 175–185 (1981)
14. Pang J. S.: Necessary and sufficient conditions for the convergence of iterative methods for the linear complementarity problem. J. Opt. Theory, Appl. 42, 1–17 (1984)
15. Pantazopoulos K.: Numerical Methods and Software for the Pricing of American Financial Derivatives. PhD Thesis, Department of Computer Sciences, Purdue University, West Lafayette, IN (1998)
16. Koulisianis M.D. and Papatheodorou T.S.: Improving projected successive overrelaxation method for linear complementarity problems. Appl. Numer. Math. 45, 29–40 (2003)

17. Yuan D, Song Y.: Modified AOR methods for linear complementarity problem. Appl. Math. Comput. 140, 53-67 (2003)
18. Cvetković Lj., Rapajić S.: How to improve MAOR method convergence area for linear complementarity problems. Appl. Math. Comput. 162, 577-584 (2005)
19. Li Y., Dai P.: Generalized AOR for linear complementarity problem. Appl. Math. Comput. 188, 7–18 (2007)
20. Saberi Najafi H., Edalatpanah S. A.: On the convergence regions of Generalized Accelerated Overrelaxation method for linear complementarity problems. J. Optim. Theory Appl. 156, 859–866 (2013)
21. Saberi Najafi H., Edalatpanah S. A.: Modification of iterative methods for solving linear complementarity problems. Engrg. Comput. 30 (2013), 910-923.
22. Herceg D., Cvetković Lj.: On an iterative method for a system of equations. Zb. Rad. Prir.- Mat. Fac., Ser. Mat. 20, 11-15 (1990).
23. Saberi Najafi H., Edalatpanah S. A.: Iterative methods with analytical preconditioning technique to linear complementarity problems. RAIRO-Oper. Res. 47, 59-71 (2013)
24. van Bokhoven W. M. G.: A class of linear complementarity problems is solvable in polynomial time. Department of Electrical Engineering, University of Technology, Eindhoven, Netherlands (1980)
25. Kappel N. W., Watson L. T.: Iterative algorithms for the linear complementarity problems. Int. J. Comput. Math. 19, 273–297 (1986)
26. Hadjidimos A., Tzoumas M.: Nonstationary extrapolated modulus algorithms for the solution of the linear complementarity problem. Linear Algebra Appl. 431, 197–210 (2009)
27. Bai Z.-Z.: Modulus-based matrix splitting iteration methods for linear complementarity problems. Numer. Linear Algebra Appl. 17, 917–933 (2010)
28. Dong J.-L., Jiang M.-Q.: A modified modulus method for symmetric positive-definite linear complementarity problems. Numer. Linear Algebra Appl. 16, 129–143 (2009)
29. Zhang L.-L.: Two-step modulus based matrix splitting iteration method for linear complementarity problems. Numer. Algor. 57, 83–99 (2011)
30. Hadjidimos A., Lapidakis M., Tzoumas M.: On iterative solution for the linear complementarity problem with an H_+-matrix. SIAM J. Matrix Anal. 33, 97–110 (2012)
31. Bai Z.-Z., Zhang L-L.: Modulus-based synchronous multisplitting iteration methods for linear complementarity problems. Numer. Linear Algebra Appl. 20, 425–439 (2013)
32. Bai Z.-Z., Zhang L.-L.: Modulus-based synchronous two-stage multisplitting iteration methods for linear complementarity problems. Numer. Algor. 62, 59–77 (2013)
33. Cvetković Lj., Kostić V.: A note on the convergence of the MSMAOR method for linear complementarity problems. Numer. Linear Algebra Appl. (2013) (DOI:10.1002/nla.1896)
34. Cvetković Lj., Hadjidimos A., Kostić V.: On the choice of parameters in MAOR type splitting methods for the linear complementarity problem. Numer. Algor. (2014) (DOI: 10.1007/s11075-014-9824-1)
35. Hadjidimos A.: Accelerated overrelaxation method. Math. Comput. 32, 149–157 (1978)
36. Hadjidimos A.: Successive Overrelaxation (SOR) and related methods. J. Comput. Appl. Math. 123, 177–199 (2000) (Also, in *"Numerical Analysis 2000, Volume 3, Linear Algebra - Linear Systems and Eigenvalues"*. van Dooren P. M., Hadjidimos A., van der Vorst H. A. (Eds), North Holland, Amsterdam (2000))
37. Ortega J. M., Rheinboldt W.: Iterative Solution of Nonlinear Equations in Several Space Variables. Classics in Applied Mathematics 30. SIAM, Philadelphia (2000)
38. Alanelli M., Hadjidimos A.: A new iterative criterion for $H-$matrices. SIAM J. Matrix Anal. Appl. 29, 160–176 (2006)
39. Bru Garcia R., Giménez I, Hadjidimos A.: Is $A \in C^{n,n}$ a general $H-$matrix? Linear Algebra Appl. 436, 364–380 (2012)