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The solution of the linear complementarity problem by the matrix analogue of the accelerated overrelaxation iterative method

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Abstract

The Linear Complementarity Problem (LCP), with an H_+-matrix coefficient, is solved by using the new "*(Projected) Matrix Analogue of the AOR (MAAOR)*" iterative method; this new method constitutes an extension of the "*Generalized AOR (GAOR)*" iterative method. In this work two sets of convergence intervals of the parameters involved are determined by the theories of "*Perron-Frobenius*" and of "*Regular Splittings*". It is shown that the intervals in question are better than any similar convergence intervals found so far by similar iterative methods. A deeper analysis reveals that the "best" values of the parameters involved are those of the (projected) scalar Gauss-Seidel iterative method. A theoretical comparison of the "best" (projected) Gauss-Seidel and the "best" modulus-based splitting Gauss-Seidel method is in favor of the former method. A number of numerical examples support most of our theoretical findings.

AMS (MOS) Subject Classifications: Primary 65F10

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Running Title: MAAOR iterative method for the solution of the LCP

1 Introduction

We begin with the definition of the *Linear Complementarity Problem (LCP)*.
"Given $A \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ find two nonnegative vectors $r, z \in \mathbb{R}^n$ satisfying the relations

$$r = Az + q \quad \text{and} \quad r^T z = 0.\text{"} \tag{1.1}$$

The LCP has many applications in science, engineering, economics, etc. (see, e.g., [9, 13, 31]).

The matrix A in (1.1) is assumed to be *irreducible*. If A is *reducible* the *LCP* can be split into a number of smaller LCPs which can be solved with much less computational cost. (For more details see the Appendix.)

As is known the *LCP* possesses a unique solution if and only if $A \in \mathbb{R}^{n\times n}$ is a $P-$matrix, namely, a matrix whose all its principal submatrices have positive determinants (see, e.g, [9, 13, 31]). In

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the present work we consider A to be an H_+-matrix, a notation introduced by Bai in [4], that is a real $H-$matrix with positive diagonal.

In general, there are three main classes of iterative methods for the solution of the LCP:
a) The "*projected methods*", the seed of which goes back to Christopherson's work [12] for the solution of the free-boundary problem for journal bearings (see also [36, 20]). His work was studied deeper by Cryer in [14, 15]. Then, other works followed based on the iterative solution of large sparse linear systems (see, e.g., [42, 43]). We mention those by Mangasarian [29], Ahn [1], Pang [34], Pantazopoulos [35], and Koulisianis and Papatheodorou [26], as well as three of the most recent ones by Li and Dai [27], Saberi Najafi and Edalatpanah [37] and Hadjidimos and Tzoumas [24].
b) The "*modulus algorithm*" introduced by van Bokhoven [41] and extended by Kappel and Watson [25] to "*block modulus algorithm*". In these works "*extrapolation*" was introduced to accelerate the convergence (see [23] and [22]). and
c) The "*modulus-based matrix splitting iterative methods*", particularly the "*modulus-based splitting accelerated overrelaxation (MBSAOR) iterative method*", introduced by Bai [5]. Bai's work exploited van Bokhoven's modulus algorithm [41] in two ways: i) A "*diagonal extrapolation matrix*" was introduced to accelerated the convergence and ii) The main matrix was split into the difference of two others making possible extensions of the classical iterative methods to be employed. His work [5] was the starting point for many others that followed (see, e.g., [18, 44, 22, 28, 17, 16, 49, 50]).

Since the mid 90s researchers have started using parallel methods based on multisplittings [32] (see, e.g., [3, 4, 6] and the most recent ones based mainly on (c) above [7, 8, 47, 45, 17, 46, 48]).

The outline of the rest of this work follows. In section 2, the "*matrix analogue of the AOR (MAAOR)*", introduced recently in [21], for the solution of the LCP by the "*projected methods*" is presented. In section 3, its convergence is analyzed and studied. In section 4, the theories of "*Perron-Frobenius*" and of "Regular Splittings" (see, e.g., [42]) provide sets of sufficient convergence intervals; the parameters involved are determined and the "best" of these parameters are found in the sense explained there. In section 5, numerical examples in support of our theory are worked out. Finally, in section 6, a number of remarks and a further discussion on some issues conclude our work.

2 The MAAOR method for the solution of the LCP

For the study of the *projected methods* the following definition is needed.

Definition 2.1. *Given any vector $x \in \mathbb{R}^n$, x_+ denotes the vector with components*

$$(x_+)_i = \max\{x_i, 0\} \, \forall \, i \in N := \{1, 2, \cdots, n\}.$$

Definition 2.1 yields the following properties for any $x, y \in \mathbb{R}^n$ (see, e.g., [29, 1])

$$\begin{array}{ll} i) \;\; (x+y)_+ \leq x_+ + y_+, & ii) \;\; x_+ - y_+ \leq (x-y)_+, \\ iii) \;\; |x| = x_+ + (-x)_+, & iv) \;\; x \leq y \;\Rightarrow\; x_+ \leq y_+. \end{array} \tag{2.1}$$

Using Definition 2.1, (1.1) is transformed into the equivalent form (see, e.g., [31])

$$z = \left(z - D^{-1}(Az+q)\right)_+ \quad \Leftrightarrow \quad z = \left(z - \left((I - \widetilde{L} - \widetilde{U})z + \widetilde{q}\right)\right)_+. \tag{2.2}$$

In (2.2), $A := D - L - U$, where D, $-L$, $-U$ are the diagonal, the strictly lower and the strictly upper triangular parts of A, respectively. Also, every entity $(\cdot)$ in (1.1) and in the first equation in (2.2) has been transformed into $(\widetilde{\cdot}) := D^{-1}(\cdot)$, with I being the identity matrix. Let Ω be a positive diagonal matrix and R be a diagonal matrix. Then, multiplying the original *LCP* by ΩD^{-1}, (1.1) is transformed into the equivalent form

$$\Omega\widetilde{r} = \Big((I - R\widetilde{L}) - \Big((I-\Omega) + (\Omega - R)\widetilde{L} + \Omega\widetilde{U}\Big)\Big)\, z + \Omega\widetilde{q} \ \text{ and } \ (\Omega\widetilde{r})^T z = 0. \tag{2.3}$$

The way equations (2.2) were obtained from (1.1), similarly from (2.3) it is obtained that

$$z = \Big(z - \Big(-R\widetilde{L}z + \Omega(I - \widetilde{L} - \widetilde{U})z + R\widetilde{L}z + \Omega\widetilde{q}\Big)\Big)_+ . \tag{2.4}$$

Note that if in (2.3) $R = \alpha\Omega$, $\alpha \in \mathbb{R}$, the GAOR method applied to (1.1) is obtained (see [24]).

3 Convergence of the Projected MAAOR method

For the solution of the fixed-point equation (2.4) the following Projected MAAOR iterative method is suggested

$$z^{(k+1)} = \Big(z^{(k)} - \Big(-R\widetilde{L}z^{(k+1)} + \Omega(I - \widetilde{L} - \widetilde{U})z^{(k)} + R\widetilde{L}z^{(k)} + \Omega\widetilde{q}\Big)\Big)_+ . \tag{3.1}$$

If the iterative method (3.1) converges, then

$$z^* = \Big(z^* - \Big(-R\widetilde{L}z^* + \Omega(I - \widetilde{L} - \widetilde{U})z^* + R\widetilde{L}z^* + \Omega\widetilde{q}\Big)\Big)_+ , \tag{3.2}$$

where z^* is the exact solution of (1.1). Based on (3.1) and (3.2) the statement below can be proved.

Theorem 3.1. *Any two consecutive error vectors of iterative scheme (3.1) are connected via*

$$|z^{(k+1)} - z^*| \leq G|z^{(k)} - z^*|, \tag{3.3}$$

where

$$G \equiv G_{R,\Omega} := \Big(I - |R||\widetilde{L}|\Big)^{-1} \Big(|I - \Omega| + |\Omega - R||\widetilde{L}| + |\Omega||\widetilde{U}|\Big) \geq 0 \tag{3.4}$$

and a sufficient condition for the Projected MAAOR iterative method to converge is $\rho(G) < 1$, where $\rho(\cdot)$ denotes spectral radius.

Proof: Using properties (2.1), then from (3.1) and (3.2), we successively obtain [1]

$$\begin{aligned}
z^{(k+1)} - z^* = {} & \Big(z^{(k)} - \Big(-R\widetilde{L}z^{(k+1)} + \Omega(I - \widetilde{L} - \widetilde{U})z^{(k)} + R\widetilde{L}z^{(k)} + \Omega\widetilde{q}\Big)\Big)_+ \\
& - \Big(z^* - \Big(-R\widetilde{L}z^* + \Omega(I - \widetilde{L} - \widetilde{U})z^* + R\widetilde{L}z^* + \Omega\widetilde{q}\Big)\Big)_+ \overset{(ii)}{\leq} \\
& \Big((z^{(k)} - z^*) - \Big(-R\widetilde{L}(z^{(k+1)} - z^*) + (\Omega - (\Omega - R)\widetilde{L} - \Omega\widetilde{U})(z^{(k)} - z^*)\Big)\Big)_+ .
\end{aligned}$$

[1] A lower case Latin numeral over a relational operator, as, e.g., "$\overset{(ii)}{\leq}$", refers to the application and/or implication of the corresponding property of (2.1).

Hence

$$
\begin{aligned}
&(z^{(k+1)}-z^*)_+ \leq \left(R\widetilde{L}(z^{(k+1)}-z^*) + \left(I-\Omega+(\Omega-R)\widetilde{L}+\Omega\widetilde{U}\right)(z^{(k)}-z^*)\right)_+ \overset{(i)}{\leq} \\
&\left(R\widetilde{L}(z^{(k+1)}-z^*)\right)_+ + \left(\left(I-\Omega+(\Omega-R)\widetilde{L}+\Omega\widetilde{U}\right)(z^{(k)}-z^*)\right)_+.
\end{aligned} \tag{3.5}
$$

Similarly, we can obtain

$$
(z^*-z^{(k+1)})_+ \overset{(i)}{\leq} \left(R\widetilde{L}(z^*-z^{(k+1)})\right)_+ + \left(\left(I-\Omega+(\Omega-R)\widetilde{L}+\Omega\widetilde{U}\right)(z^*-z^{(k)})\right)_+. \tag{3.6}
$$

Then, from (3.5) and (3.6) we get

$$
\begin{aligned}
&\left|z^{(k+1)}-z^*\right| \overset{(iii)}{=} (z^{(k+1)}-z^*)_+ + (z^*-z^{(k+1)})_+ \leq \\
&\left(R\widetilde{L}(z^{(k+1)}-z^*)\right)_+ + \left(\left(I-\Omega+(\Omega-R)\widetilde{L}+\Omega\widetilde{U}\right)(z^{(k)}-z^*)\right)_+ + \\
&\left(R\widetilde{L}(z^*-z^{(k+1)})\right)_+ + \left(\left(I-\Omega+(\Omega-R)\widetilde{L}+\Omega\widetilde{U}\right)(z^*-z^{(k)})\right)_+ \overset{(iii)}{\leq} \\
&\left|R\widetilde{L}(z^{(k+1)}-z^*)\right| + \left|\left(I-\Omega+(\Omega-R)\widetilde{L}+\Omega\widetilde{U}\right)(z^{(k)}-z^*)\right| \leq \\
&|R||\widetilde{L}||z^{(k+1)}-z^*| + \left(|I-\Omega| + |(\Omega-R)||\widetilde{L}| + |\Omega||\widetilde{U}|\right)|z^{(k)}-z^*|.
\end{aligned} \tag{3.7}
$$

From the leftmost and rightmost expressions of (3.7) we take

$$
\left(I-|R||\widetilde{L}|\right)|z^{(k+1)}-z^*| \leq \left(|I-\Omega| + |(\Omega-R)||\widetilde{L}| + |\Omega||\widetilde{U}|\right)|z^{(k)}-z^*|. \tag{3.8}
$$

Since $\rho(|R||\widetilde{L}|) = 0$, the matrix $I - |R||\widetilde{L}|$ is invertible and possesses a nonnegative Neumann expansion. Therefore,

$$
|z^{(k+1)}-z^*| \leq \left(I-|R||\widetilde{L}|\right)^{-1}\left(|I-\Omega| + |(\Omega-R)||\widetilde{L}| + |\Omega||\widetilde{U}|\right)|z^{(k)}-z^*|, \tag{3.9}
$$

and relation (3.3) is obtained showing that a sufficient condition for the Projected MAAOR iterative method to converge is $\rho(G) < 1$. □

From Theorem 3.1 and especially relation (3.3) the following corollary is obtained which was proved very useful in Theorem 3.1 of [27] and Theorem 3.1 of [24].

Corollary 3.1. *Under the notation and assumptions of Theorem 3.1, from relation (3.3) the following inequality is readily obtained*

$$
\|z^{(k+1)}-z^*\|_\infty \leq \|G\|_\infty \|z^{(k)}-z^*\|_\infty. \tag{3.10}
$$

An alternative theorem to Theorem 3.1 that gives equivalent results follows.

Theorem 3.2. *Let $z^{(k+1)}, z^{(k)}, z^{(k-1)}$, $k = 1, 2, 3\cdots$, be three successive approximations to the exact solution z^* of (3.2). Then, there holds*

$$
|z^{(k+1)}-z^{(k)}| \leq G|z^{(k)}-z^{(k-1)}|, \ k = 1, 2, 3, \cdots. \tag{3.11}
$$

Proof: Using equation (3.1) at the previous step, that is

$$z^{(k)} = \left(z^{(k-1)} - \left(-R\widetilde{L}z^{(k)} + \Omega(I - \widetilde{L} - \widetilde{U})z^{(k-1)} + R\widetilde{L}z^{(k-1)} + \Omega\widetilde{q}\right)\right)_+, \tag{3.12}$$

subtract it from (3.1), and follow step by step the proof of Theorem 3.1, (3.11) is obtained. □

Theorem 3.3. *Under the assumption that the matrix G of (3.4) satisfies $\rho(G) < 1$ ($G \geq 0$), then by the "Contraction Mapping Theorem" (see, e.g., Ortega and Rheinboldt [33]), (3.3) and (3.11) imply*

$$\left|z^{(k)} - z^*\right| \leq (I - G)^{-1}G^k\left|z^{(1)} - z^{(0)}\right|. \tag{3.13}$$

Proof: Beginning with $\left|z^{(k)} - z^*\right|$ and using relation (3.3) we successively have

$$\left|z^{(k)} - z^*\right| \leq G\left|z^{(k-1)} - z^*\right| = G\left(\left|z^{(k-1)} - z^{(k)} + z^{(k)} - z^*\right|\right) \leq G\left|z^{(k-1)} - z^{(k)}\right| + G\left|z^{(k)} - z^*\right|,$$

from which we take

$$(I - G)\left|z^{(k)} - z^*\right| \leq G\left|z^{(k)} - z^{(k-1)}\right|. \tag{3.14}$$

Since G is nonnegative and convergent, $(I - G)^{-1} \geq 0$. So, multiplying both members of (3.14) by $(I - G)^{-1}$, using (3.11), and then induction, (3.13) is obtained. □

Corollary 3.2. *Under the assumptions of Theorem 3.3, the spectral radius of the matrix coefficient in relation (3.13) is*

$$\rho\left((I - G)^{-1}G^k\right) = \frac{\rho^k(G)}{1 - \rho(G)}. \tag{3.15}$$

Proof: First we prove that $\rho\left((I - G)^{-1}G\right) = \frac{\rho(G)}{1-\rho(G)}$ and then our proof follows step by step the analysis in the bottom half of page 95 and the end part of the proof of Theorem 3.29 of Varga [42]. To avoid unnecessary repetitions we simply say that from the relation just proved, the expression in (3.15) readily follows. □

4 Convergence intervals of the parameters involved

4.1 Strictly diagonally dominant H_+-matrices

We observe that the matrix G in (3.4) and the matrix G in (4.1) of [21] are identical. So, if our matrix A is "*strictly diagonally dominant (SDD) by rows*" with positive diagonal, then sufficient conditions for G to converge are those of Theorem 4.1 of [21] depicted in Table 1.

Case	ω_i	r_i
(I)	$(0,\ 1]$	$\left(-\frac{\omega_i(1-\widetilde{l}_i-\widetilde{u}_i)}{2\widetilde{l}_i},\ \frac{\omega_i(1+\widetilde{l}_i-\widetilde{u}_i)}{2\widetilde{l}_i}\right)$
(II)	$\left[1,\ \frac{2}{1+\widetilde{l}_i+\widetilde{u}_i}\right)$	$\left(-\frac{2-\omega_i(1+\widetilde{l}_i+\widetilde{u}_i)}{2\widetilde{l}_i},\ \frac{2-\omega_i(1-\widetilde{l}_i+\widetilde{u}_i)}{2\widetilde{l}_i}\right)$

Table 1: Sufficient convergence intervals for ω_i, $i \in N$, and r_i, $i \in N\backslash\{1\}$

Some issues in connection with Table 1 should be made clear.
i) $\widetilde{l}_i, \widetilde{u}_i$, $\forall\, i \in N$, are the row sums of the matrices $\widetilde{L}$ and $\widetilde{U}$, respectively. Specifically,

$$\widetilde{l}_i = \sum_{j=1}^{i-1} |\widetilde{l}_{ij}| \;\; \forall\, i \in N\backslash\{1\} \quad \text{and} \quad \widetilde{u}_i = \sum_{j=i+1}^{n} |\widetilde{u}_{ij}| \;\; \forall\, i \in N\backslash\{n\} \quad \text{with} \quad \widetilde{l}_1 = \widetilde{u}_n = 0. \tag{4.1}$$

Whenever $\widetilde{l}_i = 0$ appears as a denominator, we may assume that $\widetilde{l}_i \to 0^+$, and the corresponding fraction tends to $-\infty$ or $+\infty$ depending on the sign of the numerator.
ii) A is irreducible and so is $\widetilde{A}$. Therefore, in Case II all ω_i's but at least one can assume the right end values of the corresponding open intervals. Similarly, in Cases I and II for all r_i's, $i \in N\backslash\{1\}$, but one can assume either the left or the right end value of the interval.
iii) In the expression for the matrix G in (3.4), the matrix R multiplies the matrix $|\widetilde{L}|$ from the left. Since $\widetilde{l}_1 = 0$, G is independent of r_1 which can be any real number. and
iv) If A is an H_+-matrix but not an SDD one, then it can be transformed into an SSD matrix by using the Algorithm in [2] for A irreducible or the Algorithm in [11] for A irreducible or reducible.

4.2 H_+-matrices

4.2.1 Introduction

In this section, we assume that A is an H_+-matrix not necessarily an SDD one and we find sufficient conditions for an upper bound of the spectral radius $\rho(G)$ of the nonnegative matrix G in (3.3)-(3.4) to be strictly less that unity. Then, the convergence of the MAAOR method for the solution of the LCP will be guaranteed. The main tools in our analysis are the theory of *Perron-Frobenius* together with that of *regular splittings* [42], *nonnegative splittings* [10], and *$M-$splittings* [38]. The definitions for the three splittins are given below.

Definition 4.1. *Let $A, M, N \in \mathbb{R}^{n\times n}$ and A and M be nonsingular. Then $A = M - N$ is:*

1. *A regular splitting of the matrix A if $M^{-1} \geq 0$ and $N \geq 0$. (For a regular splitting there holds $\rho(M^{-1}N) < 1$.)*

2. *A nonnegative splitting of the matrix A if $M^{-1}N \geq 0$. (A nonnegative splitting does* ***not*** *always imply convergence of $M^{-1}N$.)*

3. *An $M-$splitting if A and M are $M-$matrices and $N \geq 0$. (For an $M-$splitting there holds $\rho(M^{-1}N) < 1$ since it is a particular case of a regular splitting.)*

Now, observe that the matrix G in (3.4) comes from the splitting

$$\widehat{A} = (I - |R||\widetilde{L}|) - (|I - \Omega| + |\Omega - R||\widetilde{L}| + \Omega|\widetilde{U}|). \tag{4.2}$$

The diagonal and the off-diagonal parts of $I - |R||\widetilde{L}|$ are positive and nonpositive, respectively. Since $\rho(|R||\widetilde{L}|) = 0 < 1$ the matrix $I - |R||\widetilde{L}|$ is an $M-$matrix. In addition, $|I - \Omega| + |\Omega - R||\widetilde{L}| + |\Omega||\widetilde{U}| \geq 0$. Therefore, the splitting of $\widehat{A}$ will be an $M-$splitting if $\widehat{A}$ is a nonsingular $M-$matrix. Writing $\widehat{A}$ in (4.2) as

$$\widehat{A} = (I - |I - \Omega|) - \Big((|R| + |\Omega - R|)\,|\widetilde{L}| + \Omega|\widetilde{U}|\Big), \tag{4.3}$$

$\widehat{A}$ has its off-diagonal part nonpositive so the matrix $\widehat{A}$ will be an $M-$matrix and the splitting in (4.3) will be an $M-$splitting if and only if

$$\begin{aligned} &\mathrm{diag}\,(I-|I-\Omega|)>0 \Leftrightarrow \omega_i\in(0,\,2)\,\forall\,i\in N \;\; \textit{and} \\ &\rho\left((I-|I-\Omega|)^{-1}\left((|R|+|\Omega-R|)\,|\widetilde{L}|+\Omega|\widetilde{U}|\right)\right)<1. \end{aligned} \tag{4.4}$$

To go on with our analysis we must get rid of the moduli in the diagonal matrices of the spectral radius in (4.4) and so we have to consider the signs of r_i, ω_i-r_i, $1-\omega_i$, noting that for r_i we will always assume that $i\in N\backslash\{1\}$. Hence, we distinguish the following six cases:

$$\begin{array}{llllll} i) & r_i\le 0<\omega_i\le 1, & ii) & r_i\le 0,\,1\le\omega_i<2, & iii) & 0\le r_i\le\omega_i\le 1, \\ iv) & 0\le r_i\le\omega_i,\,1\le\omega_i<2, & v) & \omega_i\le r_i,\;0<\omega_i\le 1 & vi) & \omega_i\le r_i,\;1\le\omega_i<2, \end{array} \tag{4.5}$$

which will be investigated further in the next section in order to find sufficient condition intervals of the parameters ω_i, r_i for an upper bound ("*majorizer*") of the matrix G in (3.4) to converge.

4.2.2 Sufficient convergence conditions

Under the conditions in (4.4), $\widehat{A}$ in (4.3) is an $M-$matrix and can also be written as

$$\widehat{A}=(I-|I-\Omega|)\left(I-\left(\Omega^{-1}-|\Omega^{-1}-I|\right)^{-1}\left(\left(\Omega^{-1}|R|+|I-\Omega^{-1}R|\right)|\widetilde{L}|+\widetilde{U}|\right)\right). \tag{4.6}$$

Observe that the second factor above is an $M-$matrix and note that

$$\Omega^{-1}|R|+|I-\Omega^{-1}R|\ge\Omega^{-1}|R|+I-\Omega^{-1}|R|=I\ge 0.$$

Then, by the Perron-Frobenius theory, $\widehat{A}$ will still be an $M-$matrix if instead of conditions (4.4) we consider the sufficient ones

$$\omega_i\in(0,2)\,\forall\,i\in N \text{ and } \rho\left(\left(\Omega^{-1}-|\Omega^{-1}-I|\right)^{-1}\left(\Omega^{-1}|R|+|I-\Omega^{-1}R|\right)\left(|\widetilde{L}|+\widetilde{U}|\right)\right)<1. \tag{4.7}$$

Clearly, under the conditions (4.7) we have that

$$G\le\left(\Omega^{-1}-|\Omega^{-1}-I|\right)^{-1}\left(\Omega^{-1}|R|+|I-\Omega^{-1}R|\right)\left(|\widetilde{L}|+\widetilde{U}|\right) \tag{4.8}$$

and appealing once again to the Perron-Frobenius theory we obtain that

$$\rho(G)\le\rho\left(\left(\Omega^{-1}-|\Omega^{-1}-I|\right)^{-1}\left(\Omega^{-1}|R|+|I-\Omega^{-1}R|\right)\left(|\widetilde{L}|+\widetilde{U}|\right)\right)<1. \tag{4.9}$$

From (4.6)-(4.7) we can readily obtain the statement below.

Theorem 4.1. *Sufficient conditions for convergence of the MAAOR method for the solution of the LCP, with an irreducible H_+-matrix, are the following*

$$\max_{i\in N}\frac{\frac{|r_i|}{\omega_i}+\left|1-\frac{r_i}{\omega_i}\right|}{\frac{1}{\omega_i}-\left|\frac{1}{\omega_i}-1\right|}=\max_i\frac{|r_i|+|\omega_i-r_i|}{1-|1-\omega_i|}\le\frac{1}{\rho(|\widetilde{B}|)}\quad\forall\,\omega_i\in(0,2), \tag{4.10}$$

with strict inequality for at least one i. Also, the matrix on the right side of (4.8) is an $M-$matrix.

Let us assume that all six cases in (4.5) are present in (4.10). Below we analyze only Case (i). All other cases are analyzed in a similar way and the results obtained are summarized in Table 2.

i) If $r_i \leq 0 < \omega_i \leq 1$, then

$$\frac{\omega_i - 2r_i}{\omega_i} \leq \frac{1}{\rho(|\widetilde{B}|)} \iff \frac{\omega_i\left(\rho(|\widetilde{B}|) - 1\right)}{2\rho(|\widetilde{B}|)} \leq r_i \leq 0 < \omega_i \leq 1.$$

Case	Sufficient convergence intervals
(i)	$\frac{\omega_i\left(\rho(\lvert\widetilde{B}\rvert)-1\right)}{2\rho(\lvert\widetilde{B}\rvert)} \leq r_i \leq 0 < \omega_i \leq 1$
(ii)	$\frac{\omega_{i_2}\left(1+\rho(\lvert\widetilde{B}\rvert)\right)-2}{2\rho(\lvert\widetilde{B}\rvert)} \leq r_i \leq 0$ and $1 \leq \omega_i \leq \frac{2}{1+\rho(\lvert\widetilde{B}\rvert)}$
(iii)	$0 \leq r_i \leq \omega_i \leq 1 < \left(\frac{1}{\rho(\lvert\widetilde{B}\rvert)}\right)$
(iv)	$0 \leq r_i \leq \omega_i$ and $1 \leq \omega_i \leq \frac{2}{1+\rho(\lvert\widetilde{B}\rvert)}$
(v)	$0 < \omega_i \leq r_i \leq \frac{\omega_i\left(1+\rho(\lvert\widetilde{B}\rvert)\right)}{2\rho(\lvert\widetilde{B}\rvert)}$ and $\omega_i \leq 1$
(vi)	$1 \leq \omega_i \leq r_i \leq \frac{2-\omega_i\left(1-\rho(\lvert\widetilde{B}\rvert)\right)}{2\rho(\lvert\widetilde{B}\rvert)}$ and $\omega_i \leq \frac{2}{1+\rho(\lvert\widetilde{B}\rvert)}$

Table 2: Sufficient convergence intervals for $\omega_i \, \forall\, i \in N$ and $r_i \,\forall\, i \in N \backslash \{1\}$

Note: In a certain case at least one of the inequalities coming from relations (4.10) must be strict

4.2.3 "Best" MAAOR iterative method

In this section we find the "best" MAAOR iterative method for the solution of the LCP in the sense that we make the majorizer of the matrix G in (4.8), and hence its spectral radius in the middle of relations (4.9), be as small as possible. For this we assume that the diagonal elements of R and Ω satisfy the sufficient convergence conditions of all six Cases of Table 2. Let then

$$\begin{aligned} R &= \operatorname{diag}\left(\operatorname{diag}(R_{(i)}), \operatorname{diag}(R_{(ii)}), \operatorname{diag}(R_{(iii)}), \operatorname{diag}(R_{(iv)}), \operatorname{diag}(R_{(v)}), \operatorname{diag}(R_{(vi)})\right), \\ \Omega &= \operatorname{diag}\left(\operatorname{diag}(\Omega_{(i)}), \operatorname{diag}(\Omega_{(ii)}), \operatorname{diag}(\Omega_{(iii)}), \operatorname{diag}(\Omega_{(iv)}), \operatorname{diag}(\Omega_{(v)}), \operatorname{diag}(\Omega_{(vi)})\right), \end{aligned} \tag{4.11}$$

where

$$R_j, \Omega_j \in \mathbb{R}^{\operatorname{card}(j)\times \operatorname{card}(j)} \quad \forall\, j \in \{(i),\, (ii),\, \cdots,\, (vi)\},$$

with $\sum_{j=(i)}^{(vi)} \operatorname{card}(j) = n$, have elements satisfying all the sufficient convergence conditions of Cases $(i) - (vi)$ of Table (2), respectively. Note that if the diagonal elements of R and Ω are not in the above sequence then a similarity permutation of the original LCP can make them be.

First, the "best" case out of Cases (i) and (ii) is determined.

a) Let the $M-$splitting of the matrix $G = M - N$ leading to

$$M^{-1} = I + |R||\widetilde{L}| + (|R||\widetilde{L}|)^2 + \cdots + (|R||\widetilde{L}|)^{n-1} \geq 0, \quad N = |I - \Omega| + |\Omega - R||\widetilde{L}| + \Omega|\widetilde{U}| \geq 0$$

and let below the auxiliary nonnegative splitting corresponding to $R_{(i)} = 0$ and $R_{(ii)} = 0$, namely

$$\begin{gathered} M^{-1}_{R_{(i)}=0, R_{(ii)}=0} = \\ I + |R|_{R_{(i)}=0,R_{(ii)}=0}|\widetilde{L}| + (|R|_{R_{(i)}=0,R_{(ii)}=0}|\widetilde{L}|)^2 + \cdots + (|R|_{R_{(i)}=0,R_{(ii)}=0}|\widetilde{L}|)^{n-1} \geq 0, \\ N_{R_{(i)}=0,R_{(ii)}=0} = |I - \Omega| + |\Omega - R|_{R_{(i)}=0,R_{(ii)}=0}|\widetilde{L}| + \Omega|\widetilde{U}| \geq 0. \end{gathered}$$

Clearly,

$$0 \leq M^{-1}_{R_{(i)}=0,R_{(ii)}=0} \leq M^{-1} \quad \text{and} \quad 0 \leq N_{R_{(i)}=0,R_{(ii)}=0} \leq N$$

leading, by the Perron-Frobenius theory, to

$$\rho\left(M^{-1}_{R_{(i)}=0,R_{(ii)}=0} N_{R_{(i)}=0,R_{(ii)}=0}\right) \leq \rho\left(M^{-1}N\right) < 1.$$

Hence, the particular case $R_{(i)} = 0, R_{(ii)} = 0$, corresponding to two extreme cases of Cases (i) and (ii), gives a better spectral radius than the general Cases (i) and (ii) do. This suggests to incorporate the "best" Cases (i) and (ii) into the Cases (iii) and Case (iv), respectively. This incorporation can be accomplished by a similarity permutation.

Next, the "best" case out of the "new" Cases $(iii) - (vi)$ is determined.

b) Consider Cases (v) and (vi) and assume that Cases (i) and (ii) have already been incorporated into Cases (iii) and (iv), respectively, with $R_{(i)} = 0, R_{(ii)} = 0$. So, the "new" Cases $(iii') = (i) \cup (iii)$ and $(iv') = (ii) \cup (iv)$. Recall that the "best" splitting found so far is $\widehat{A}' = M' - N'$, where

$$M' = \left[\begin{array}{c|c|c} I_j & & \\ \hline & I_{(v)} & \\ \hline & & I_{(vi)} \end{array}\right] - \left[\begin{array}{c|c|c} R_j & & \\ \hline & R_{(v)} & \\ \hline & & R_{(vi)} \end{array}\right] |\widetilde{L}|,$$

$$N' = \left[\begin{array}{c|c|c} |I_j - \Omega_j| & & \\ \hline & I_{(v)} - \Omega_{(v)} & \\ \hline & & \Omega_{(vi)} - I_{(vi)} \end{array}\right] + \left[\begin{array}{c|c|c} \Omega_j - R_j & & \\ \hline & R_{(v)} - \Omega_{(v)} & \\ \hline & & R_{(vi)} - \Omega_{(vi)} \end{array}\right] |\widetilde{L}| +$$
$$\left[\begin{array}{c|c|c} \Omega_j & & \\ \hline & \Omega_{(v)} & \\ \hline & & \Omega_{(vi)} \end{array}\right] |\widetilde{U}|.$$

Together with the above splitting we consider the auxiliary splitting $\widehat{A}'' = M'' - N''$, where

$$M'' = \left[\begin{array}{c|c|c} I_j & & \\ \hline & I_{(v)} & \\ \hline & & I_{(vi)} \end{array}\right] - \left[\begin{array}{c|c|c} R_j & & \\ \hline & \Omega_{(v)} & \\ \hline & & \Omega_{(vi)} \end{array}\right] |\widetilde{L}|,$$

$$N'' = \left[\begin{array}{c|c|c} |I_j - \Omega_j| & & \\ \hline & I_{(v)} - \Omega_{(v)} & \\ \hline & & \Omega_{(vi)} - I_{(vi)} \end{array}\right] + \left[\begin{array}{c|c|c} \Omega_j - R_j & & \\ \hline & 0_{(v)} & \\ \hline & & 0_{(vi)} \end{array}\right] |\widetilde{L}| +$$
$$\left[\begin{array}{c|c|c} \Omega_j & & \\ \hline & \Omega_{(v)} & \\ \hline & & \Omega_{(vi)} \end{array}\right] |\widetilde{U}|, \quad j = (iii') \cup (iv').$$

Since the two $M-$matrices M' and M'' satisfy $M' \leq M''$ it is implied that $0 \leq M''^{-1} \leq M'^{-1}$; it is also $0 \leq N'' \leq N'$. These inequalities lead to

$$\rho\left(M''^{-1}N''\right) \leq \rho\left(M'^{-1}N'\right) < 1.$$

Hence, the "best" splitting of the two splittins considered is the latter one $\widehat{A}'' = M'' - N''$.

c) Then, together with the above "best" splitting $\widehat{A}''(R,\Omega) = M''(R,\Omega) - N''(R,\Omega)$ we consider the splitting $\widehat{A}''(\Omega,\Omega) = M''(\Omega,\Omega) - N''(\Omega,\Omega)$, where R_j is replaced by Ω_j. It is seen that

$$M''^{-1}(\Omega,\Omega) \geq M''^{-1}(R,\Omega) \geq 0, \quad 0 \leq N''(\Omega,\Omega) \leq N''(R,\Omega), \tag{4.12}$$

because the $M-$matrices $M''(\Omega,\Omega)$ and $M''(R,\Omega)$ satisfy $M''(\Omega,\Omega) \leq M''(R,\Omega)$. Therefore, a direct comparison of the spectral radii as in the previous cases (a) and (b) can not be made. However, we observe that in both cases the difference of the relevant M's and N's produce the same $M-$matrix

$$\begin{aligned} \widehat{A}''(R,\Omega) = \widehat{A}''(\Omega,\Omega) = (I - |I-\Omega|) - \Omega|\widetilde{B}| = \\ (I - |I-\Omega|)\left(I - (\Omega^{-1} - |\Omega^{-1} - I|)^{-1}|\widetilde{B}|\right). \end{aligned} \tag{4.13}$$

In view of the relation $N''(\Omega,\Omega) \leq N''(R,\Omega)$ above we will obtain

$$\rho\left(\widehat{A}''^{-1}(\Omega,\Omega)N''(\Omega,\Omega)\right) \leq \rho\left(\widehat{A}''^{-1}(R,\Omega)N''(R,\Omega)\right), \tag{4.14}$$

consequently, by Theorem 3.29 of Varga [42] the following two sets of relations are obtained

$$\rho\left(M''^{-1}(\Omega,\Omega)N''(\Omega,\Omega)\right) = \frac{\rho\left(\widehat{A}''^{-1}(\Omega,\Omega)N''(\Omega,\Omega)\right)}{1+\rho\left(\widehat{A}''^{-1}(\Omega,\Omega)N''(\Omega,\Omega)\right)} < 1, \tag{4.15}$$

$$\rho\left(M''^{-1}(R,\Omega)N''(R,\Omega)\right) = \frac{\rho\left(\widehat{A}''^{-1}(R,\Omega)N''(R,\Omega)\right)}{1+\rho\left(\widehat{A}''^{-1}(R,\Omega)N''(R,\Omega)\right)} < 1. \tag{4.16}$$

From relations (4.12)-(4.16) it is concluded that

$$0 \leq \rho\left(M''^{-1}(\Omega,\Omega)N''(\Omega,\Omega)\right) \leq \rho\left(M''^{-1}(R,\Omega)N''(R,\Omega)\right) < 1. \tag{4.17}$$

Now, since the two matrices in (4.13) are $M-$matrices sufficient and necessary convergence conditions are

$$0 < \omega_i \leq \frac{2}{1+\rho(|\widetilde{B}|)} \quad \text{and} \quad 0 \leq r_i \leq \omega_i \ \ \forall\, i \in (iii') \cup (iv'), \tag{4.18}$$

with strict inequality in the first set of relations for at least one i. As a by-product of the analysis so far we have that

Theorem 4.2. *Conditions (4.18) constitute sufficient conditions for the MAAOR method for the solution of the LCP, with an irreducible H_+-matrix, to converge.*

d) By a new similarity permutation we bring Cases $(iii'') = (iii') \cup (v)$ as they have been modified to the first three block positions and Cases $(iv'') = (iv') \cup (vi)$ to the last three. Hence, under conditions (4.18), the matrices of the "best" splitting so far can be written as

$$M''' = \left[\begin{array}{c|c} I_j & \\ \hline & I_k \end{array}\right] - \left[\begin{array}{c|c} \Omega_j & \\ \hline & \Omega_k \end{array}\right]|\widetilde{L}|, \quad N''' = \left[\begin{array}{c|c} I_j - \Omega_j & \\ \hline & \Omega_k - I_k \end{array}\right] + \left[\begin{array}{c|c} \Omega_j & \\ \hline & \Omega_k \end{array}\right]|\widetilde{U}|,$$

where $j = (iii'')$ and $k = (iv'')$. Let $\widehat{A}''' = M''' - N'''$ be written as

$$\begin{aligned}\widehat{A}''' &= \left[\begin{array}{c|c}\Omega_j & \\ \hline & 2I_k - \Omega_k\end{array}\right]\left(I - \left[\begin{array}{c|c}\Omega_j & \\ \hline & 2I_k - \Omega_k\end{array}\right]^{-1}\left[\begin{array}{c|c}\Omega_j & \\ \hline & \Omega_k\end{array}\right](|\widetilde{L}| + |\widetilde{U}|)\right) \\ &= \left[\begin{array}{c|c}\Omega_j & \\ \hline & 2I_k - \Omega_k\end{array}\right]\left(I - \left[\begin{array}{c|c}I_j & \\ \hline & (2\Omega_k^{-1} - I_k)^{-1}\end{array}\right]|\widetilde{B}|\right).\end{aligned}$$

From the above expression, $\widehat{A}'''$ is an $M-$matrix if $\max\left\{\rho(|\widetilde{B}|), \max_k\left(\frac{1}{\frac{2}{\omega_k}-1}\right)\rho(|\widetilde{B}|)\right\} \le 1$ or, equivalently, $\max\left\{1, \frac{\max\omega_k}{2-\max\omega_k}\right\} \le \frac{1}{\rho(|\widetilde{B}|)}$ implying, eventually, that $0 < \omega_j \le 1$ and $1 \le \omega_k \le \frac{2}{1+\rho(|\widetilde{B}|)}$, with at least one strict inequality on the right in the second set of relations in case $k = N$.

Remark 4.1. *As a by-product of the analysis and the results just found we have that for H_+-matrices A and for $R = \Omega$ sufficient conditions for the "Modified SOR (MSOR)" iterative method for the solution of the LCP to converge are that conditions (4.18) become*

$$0 < \omega_i \le \frac{2}{1+\rho(|\widetilde{B}|)} \quad \forall\, i \in N, \tag{4.19}$$

with strict inequality for at least one i (see also Song [39]).

e) Now, consider the new auxiliary splitting

$$M_1''' = \left[\begin{array}{c|c}\Omega_j & \\ \hline & \Omega_k\end{array}\right] - \left[\begin{array}{c|c}\Omega_j & \\ \hline & \Omega_k\end{array}\right]|\widetilde{L}|, \quad N_1''' = \left[\begin{array}{c|c}\Omega_j & \\ \hline & \Omega_k\end{array}\right]|\widetilde{U}|. \tag{4.20}$$

and let

$$\widehat{A}_1''' = M_1''' - N_1''', \quad \widehat{A}_2''' \equiv \widehat{A}''' = M''' - N''' \equiv M_2''' - N_2'''.$$

Clearly, both splittings are *nonnegative splittings* since $M_i'''^{-1}N_i''' \ge 0, \quad i = 1, 2$. Also,

$$\begin{aligned}\widehat{A}_1'''^{-1} &= (I - |\widetilde{B}|)^{-1}\left[\begin{array}{c|c}\Omega_j & \\ \hline & \Omega_k\end{array}\right]^{-1}, \\ \widehat{A}_2'''^{-1} &= \left(I - \left[\begin{array}{c|c}I_j & \\ \hline & (2\Omega_k^{-1} - I_k)^{-1}\end{array}\right]|\widetilde{B}|\right)^{-1}\left[\begin{array}{c|c}\Omega_j & \\ \hline & 2I_k - \Omega_k\end{array}\right]^{-1}.\end{aligned} \tag{4.21}$$

It is $\frac{1}{\rho(|\widetilde{B}|)}I_k \ge \operatorname{diag}\left((2\Omega_k^{-1} - I_k)^{-1}\right) \ge \operatorname{diag}(I_k)$ and $\operatorname{diag}\left((2I_k - \Omega_k)^{-1}\right) \ge \operatorname{diag}\left(\Omega_k^{-1}\right)$. These relations imply that each matrix $I - |\widetilde{B}|$ and $I - \left[\begin{array}{c|c}I_j & \\ \hline & (2\Omega_k^{-1} - I_k)^{-1}\end{array}\right]|\widetilde{B}|$ is an $M-$matrix and the splittings considered are regular splittings of $M-$matrices and so they are convergent. Since each factor of $\widehat{A}_2'''^{-1}$ is nonnegative and greater than or equal to the corresponding nonnegative factor of $\widehat{A}_1'''^{-1}$ we obtain that

$$\widehat{A}_2'''^{-1} \ge \widehat{A}_1'''^{-1} \ge 0.$$

If we denote by x_i, $i = 1, 2$, the *Perron vectors* of the nonnegative convergent matrices $M_i'''^{-1}N_i'''$, $i = 1, 2$, respectively, we have that $N_2'''x_1 \ge N_1'''x_1 \ge 0$, with $N_i''' \ge 0$, $i = 1, 2$, then all the assumptions of Theorem 3.13 by Marek and Szyld [30] hold and, therefore,

$$\rho\left(M_1'''^{-1}N_1'''\right) \le \rho\left(M_2'''^{-1}N_2'''\right) < 1. \tag{4.22}$$

Consequently, we have just proved that

Theorem 4.3. *By relations (4.22) the splitting $\widehat{A}_1''' = M_1''' - N_1'''$ (4.20) is the "best" of all possible convergent splittings for the solution of the LCP and is nothing but the Projected Gauss-Seidel splitting of the matrix $\widehat{A}$, namely*

$$M_1''' - N_1''' = (I - |\widetilde{L}|) - |\widetilde{U}|.$$

Corollary 4.1. *Since the matrix A of LCP in (1.1) is irreducible, so is the matrix $|\widetilde{B}|$, meaning that ${\widehat{A}_2'''}^{-1} > 0$. Then, if at least one diagonal element of either Ω_j or Ω_k in (4.21) is different from 1, then $N_1 \neq N_2$ and, by Theorem 3.13 by Marek and Szyld [30], the inequality in (4.22) is strict.*

4.2.4 MAAOR versus MBSAOR

In this section we make a theoretical comparison of the "*majorizers*" of the present method and of the well-known "*Modulus-Based Splitting Accelerated Overrelaxation (MBSAOR)*" iterative method for the solution of the *LCP* [5] when the matrix coefficient A is an H_+-matrix. In a series of papers [44, 22, 17, 16], the original convergence intervals for the parameters α, β, used in [5], were successively widened. In only two of these works [22, 16], the "*best*" *MBSAOR*, in the sense of minimizing the corresponding majorizer, was found and it was the same Gauss-Seidel (MBSGS) method, despite some minor differences in these two works. Their "best" iteration matrix was given by

$$\widehat{T}_1 = (2I - |\widetilde{L}|)^{-1}(|\widetilde{L}| + 2|\widetilde{U}|). \tag{4.23}$$

In the present work it has been found that the "*best*" Projected *MAAOR* iterative method is again the Projected Gauss-Seidel (MAGS) iterative method. Its "best" iteration matrix is

$$\widehat{T}_2 = (I - |\widetilde{L}|)^{-1}|\widetilde{U}|. \tag{4.24}$$

However, it should be reminded that from (3.3)-(3.4) and (4.8) the operator G and, therefore, T_2 is a majorizer of an unknown operator T_2' satisfying $|z^{(k+1)} - z^*| = T_2'|z^{(k)} - z^*|$. Specifically,

$$0 \leq T_2' \leq G \leq \left(\Omega^{-1} - |\Omega^{-1} - I|\right)^{-1} \left(\Omega^{-1}|R| + |I - \Omega^{-1}R|\right) \left(|\widetilde{L}| + \widetilde{U}|\right) =: T_2.$$

Similarly, the operator T_1 is a majorizer of the operator $\widehat{\mathcal{L}}_\Omega$ (see [5], relations (9)-(10)), which in turn, is a majorizer of an unknown operator T_1' such that $|x^{(k+1)} - x_*| = T_1'|x^{(k)} - x_*|$, according to the notation in [5].

Consequently, in this section only a theoretical comparison of the majorizers associated with the two methods is made and so only in this sense the comparison is meant.

Note that the nonnegative matrices $\widehat{T}_1$ and $\widehat{T}_2$ come from the splittings of the $M-$matrices $\widetilde{A}_1$ and $\widetilde{A}_2$, respectively,

$$\widetilde{A}_1 = \underbrace{(2I - |\widetilde{L}|)}_{M_1} - \underbrace{(|\widetilde{L}| + 2|\widetilde{U}|)}_{N_1}, \quad \widetilde{A}_2 = \underbrace{(I - |\widetilde{L}|)}_{M_2} - \underbrace{|\widetilde{U}|}_{N_2}. \tag{4.25}$$

The splittings in (4.25) are $M-$splittings because the matrices M_1, M_2 are $M-$matrices and $N_1, N_2 \geq 0$. Observe that

$$\widetilde{A}_1^{-1}N_1 = \frac{1}{2}(I - |\widetilde{B}|)^{-1}(|\widetilde{L}| + 2|\widetilde{U}|) = (I - |\widetilde{B}|)^{-1}(\frac{1}{2}|\widetilde{L}| + |\widetilde{U}|) \geq (I - |\widetilde{B}|)^{-1}|\widetilde{U}| = \widetilde{A}_2^{-1}N_2 \geq 0.$$

Then, by Theorem 3.29 of Varga [42] there holds

$$\rho(\widehat{T}_2) \leq \rho(\widehat{T}_1) < 1. \tag{4.26}$$

Since the matrix A, and $\widetilde{A}$, is irreducible the inequality in (4.26) is strict. Consequently, the following statement holds.

Theorem 4.4. *The "best" of the two "best" iterative methods, the MBSGS and the Projected MAGS, is the latter one.*

Remark 4.2. *It should be pointed out once again that both "best" operators (majorizers) $\widehat{T}_1$ and $\widehat{T}_2$ constitute upper bounds of the actual operators in the corresponding methods. This is due to the nature of the analyses in [5] and in the present work since the actual operators can not be determined. So, Theorem 4.4 is of relative value although it is the only way one can use to make a theoretical comparison of the performance of the two methods.*

5 Numerical examples

Many numerical examples were run on a computer to verify: a) The successive improvement of $\rho(G)$ as the analysis of section 4.2.3 suggests. and b) The theoretical result of section 4.2.4 that the "best" Projected MAAOR (MAGS) is better than the "best" MBSAOR (MBSGS) method.

Example 5.1: Without loss of generality consider the following irreducible $M-$matrix (**not** *SDD*) with diagonal elements equal to one, so it is an H_+-matrix with the spectral radius of its Jacobi iteration matrix being $\rho(|\widetilde{B}|) = 0.9085$.

$$A = \begin{bmatrix} 1 & -0.2 & -0.2 & 0 & -0.2 & -0.2 & -0.2 \\ -0.3 & 1 & 0 & -0.2 & -0.1 & -0.1 & -0.2 \\ 0 & -0.3 & 1 & -0.2 & -0.2 & -0.1 & 0 \\ -0.3 & -0.1 & -0.3 & 1 & 0 & -0.3 & -0.1 \\ -0.2 & -0.3 & -0.2 & -0.2 & 1 & 0 & -0.1 \\ 0 & -0.3 & -0.3 & -0.1 & 0 & 1 & -0.2 \\ -0.1 & -0.1 & -0.2 & -0.1 & -0.1 & -0.1 & 1 \end{bmatrix} \in \mathbb{R}^{7\times 7}. \tag{5.1}$$

In Table 3, the successive improvement of the spectral radius $\rho(G)$ as the analysis of section 4.2.3 suggests is seen, despite the fact that some of the r_i's and ω_i's where chosen to lie outside the convergence intervals of Table 2 (see also Theorem 4.2 and Remark 4.1) in order to show the sufficiency of the determined intervals. Note that we always take $r_1 = \omega_1 = 1$.

Example(s) 5.2: A number of examples, based on the main Examples 5.1 and 5.2 of [5], were implemented in MATLAB R2009b and ran on a PC with a 3.50 GHz 64 bit processor and 4GB memory. The specifics of the LCP problems are of the general form

$$\begin{gathered} A = \widetilde{A} + \mu I_n \in \mathbb{R}^{n\times n}, \quad \widetilde{A} = I_m \otimes S + S \otimes I_m \in \mathbb{R}^{n,\times n}, \quad S = \text{tridiag}(\alpha, 2, \beta) \in \mathbb{R}^{m\times m}, \\ n = m^2, \quad (\alpha,\beta) = (\pm 1, \pm 1), (\pm 1.5, \pm 0.5), (\pm 0.5, \pm 1.5), \quad \mu = 0, 2, 4, \\ \text{with their solutions always being } \; z^* = [1\,2\,1\,2\cdots]^T \in \mathbb{R}^{n\times n} \; \text{ and } \; r^* = 0 \in \mathbb{R}^{n\times n}. \end{gathered} \tag{5.2}$$

In all the examples $q = r^* - Az^*$, $z^{(0)} = 0 \in \mathbb{R}^{n\times n}$, the stopping criterion was $||z^{(k)} - z^*||_\infty \leq 0.5 \times 10^{-15}$ and m = 10, 20, 30, 40, 50, 60 (n = 100, 400, 900, 1600, 2500, 3600). If after 5000 iterations there was no convergence this is denoted with a "$-$".

Case	diag(Ω)	diag(R)	$\rho(G)$
	$[1,\ 0.8,\ 0.8,\ 1,\ 0.9,\ 0.9,\ 1.1]$	$[1,\ -0.1,\ 0,\ 0.3,\ 0.4,\ 1,\ 1.2]$	0.9783
(a)	same as above	$[1,\ 0,\ 0,\ 0.3,\ 0.4,\ 1,\ 1.2]$	0.9610
(b)	same as above	$[1,\ 0.8,\ 0.8,\ 1,\ 0.9,\ 1,\ 1.2]$	0.9468
(c)	same as above	$[1,\ 0.8,\ 0.8,\ 1,\ 0.9,\ 0.9, 1.1]$	0.8848
(d)	$[1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1.1]$	$[1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1.1]$	0.8583
(e)	$[1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1]$	$[1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1]$	0.8160

Table 3: Successive improvement of $\rho(G)$ as this was proved in section 4.2.3.

In Table 4 the results for the Examples 5.1 and 5.2 of [5] are illustrated. Note that MBSGS denotes the methods of [22] (relations (3.1), (3.3), (3.6), with $\Omega = D$) and [16] (relations (2.1), (2.3), (3.1), with $\Omega = D$), while "iter" and "CPU" denote the number of iterations needed to satisfy the convergence criterion and the CPU time in seconds, respectively.

It should be reported that we ran all the examples for the pairs (α, β) and for the μ's in (5.2). From their CPU times it can be concluded that in almost all of the experiments the theoretical result of section 4.2.4 was verified, that is the MAGS method is better than the MBSGS one.

The MBSGS method was better than the MAGS method in the following few cases: In Example 5.2.1 of Table 4 and in those for $\mu = 0, 2$, all n and $(\alpha, \beta) = (-1, 1)$ and for $\mu = 0$, all n and $(\alpha, \beta) = (1, -1)$. Also, in Example 5.2.2 of Table 4 and in those for $\mu = 0$, all n and $(\alpha, \beta) = (-1.5, 0.5), (1.5, -0.5), (-0.5, 1.5)$. It should be said that in most of the cases and for small values of $m \approx 10$, MBSGS was better than MAGS. It is also observed that in some cases, as in Example 5.2.1 of Table 4 for $\mu = 0$, there was no convergence after 5000 iterations.

6 Concluding remarks and discussion

In this work the solution of the LCP when its matrix is an irreducible H_+-matix by the Projected MAAOR method was studied. As was proven, the convergence intervals for both matrix-parameters R and Ω extend those of the Generalized AOR (GAOR) iterative method where $R = \alpha\Omega$.

Sufficient convergence intervals were found for A being an *SDD* H_+-matrix (Table 1) and an (irreducible) H_+-matrix (Table 2). Use of the Perron-Frobenius theory for nonnegative matrices and also that of regular splittings and their extensions was make that enabled us to prove that the "best" of all Projected MAAOR methods, in the sense that has already been explained, for the solution of the LCP, with an H_+-matrix, is the Projected Gauss-Seidel method.

Since the matrix G connecting the moduli of the error vectors in the MAAOR iterative method for the solution of a (complex) linear system of [21] and the relevant matrix G of the Projected MAAOR method for the solution of the LCP with an H_+-matrix are identical, it leads us to the following observation. Let a linear system with an $H-$matrix coefficient,

$$A_1 x = b,\ A_1 \in \mathbb{C}^{n\times n},\ b \in \mathbb{C}^n,$$

which is solved by the MAAOR method, and an LCP with an H_+-matrix,

$$r = A_2 z + q, \quad A_2 \in \mathbb{R}^{n\times n},\ q \in \mathbb{R}^n,\ r^T z = 0,$$

Example 5.2.1					Example 5.2.2				
$(\alpha,\beta)=(-1,-1)$					$(\alpha,\beta)=(-1.5,-0.5)$				
$\mu=0$					$\mu=0$				
	MAGS		MBSGS			MAGS		MBSGS	
m	iter	CPU	iter	CPU	m	iter	CPU	iter	CPU
10	424	0.4688	607	0.4063	10	104	0.1875	158	0.0938
20	1522	10.4844	2141	19.1719	20	138	0.9219	219	2.0313
30	3352	87.7969	4578	224.7969	30	163	4.1250	269	13.1250
40	–	–	–	–	40	183	12.5938	308	47.7500
50	–	–	–	–	50	204	31.9844	350	132.9375
60	–	–	–	–	60	221	67.9844	403	329.1094
$\mu=2$					$\mu=2$				
	MAGS		MBSGS			MAGS		MBSGS	
m	iter	CPU	iter	CPU	m	iter	CPU	iter	CPU
10	47	0.0469	64	0.0313	10	32	0.0313	52	0.0313
20	52	0.3594	69	0.6563	20	34	0.2344	59	0.5469
30	53	1.3750	70	3.5156	30	35	0.8906	61	3.0469
40	53	3.6563	71	11.4219	40	35	2.4063	61	9.7188
50	53	8.3750	71	27.9531	50	35	5.4688	61	23.8281
60	53	16.3750	71	58.8125	60	35	10.8281	61	50.3594
$\mu=4$					$\mu=4$				
	MAGS		MBSGS			MAGS		MBSGS	
m	iter	CPU	iter	CPU	m	iter	CPU	iter	CPU
10	31	0.0313	41	0.0156	10	23	0.0156	36	0.0156
20	34	0.2344	43	0.3906	20	24	0.1875	38	0.3438
30	34	0.9219	43	2.1250	30	24	0.6250	39	1.9375
40	34	2.3906	43	6.7500	40	24	1.6250	39	6.0625
50	34	5.3750	43	16.5781	50	24	3.7813	39	14.8594
60	34	10.5469	43	34.7656	60	24	7.3750	39	31.1875

Table 4: Examples 5.2.1 and 5.2.2

solved by the Projected MAAOR method and let the two matrices have identical moduli of their associated Jacobi matrices. Then, all the sufficient convergence conditions illustrated in Tables 1 and 2 and the "best" MAAOR iterative method for both problems are exactly the same.

Illustrative numerical examples verify our theoretical findings in almost all possible cases considered. Specifically: Example 5.1 verifies the improvement of $\rho(G)$ as one follows step by step the analysis of section 4.2.3 even if the matrix-parameters R and Ω are not taken from the intervals the sufficient convergence conditions suggest (see Tables 2 and 3).

Finally, the theory of section 4.2.4 was verified in most of the numerical examples run on a computer as was explained in detail for Example(s) 5.2. However, we should add once more that, the reader having in mind the detailed analysis of section 4.2.4, much more theoretical work is needed in order to obtain stricter upper bound operators (majorizers) for the present method and the method in [5] before we decide which of the two methods is the best to be employed in a particular case. Otherwise the comparison between the two methods has to be justified by many more characteristic examples coming from real life problems. In both these directions we have been working.

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Appendix

A The reducible case

If the matrix A in (1.1) is *reducible* then a suitable similarity permutation can put A into its *Frobenius normal form* (see the articles by Tarjan [40], Duff and Reid [19], and Bru Garcia et al [11]). Assuming that $P \in \mathbb{R}^{n\times n}$ is the similarity permutation matrix, the LCP in (1.1) is transformed as follows

$$(Pr) = (PAP^T)(Pz) + (Pq) \quad \text{and} \quad (Pr)^T(Pz) = 0. \tag{A.1}$$

If we relabel the entities Pr, Pz, PAP^T, Pq as r, A, z, q, respectively, we will have that

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_i \\ \vdots \\ r_{p-1} \\ r_p \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1i} & \cdots & A_{1,p-1} & A_{1p} \\ & A_{22} & \cdots & A_{2i} & \cdots & A_{2,p-1} & A_{2p} \\ & & \ddots & \vdots & & \vdots & \vdots \\ & & & A_{ii} & \cdots & A_{i,p-1} & A_{ip} \\ & & & & \ddots & \vdots & \vdots \\ & & & & & A_{p-1,p-1} & A_{p-1,p} \\ & & & & & & A_{pp} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \\ \vdots \\ z_{p-1} \\ z_p \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_i \\ \vdots \\ q_{p-1} \\ q_p \end{bmatrix} \tag{A.2}$$
$$\text{and} \quad \textstyle\sum_{i=1}^{p} r_i^T z_i = 0,$$

where the diagonal matrices A_{ii} are $n_i \times n_i$ blocks, with $\sum_{i=1}^{p} n_i = n$, and each of the sub-vectors r_i, z_i, q_i has n_i, $i \in \{1, 2, \cdots, p\}$, components. Clearly, the relabeled matrix A is an H_+-matrix and so are all the diagonal blocks A_{ii}, $i \in \{1, 2, \cdots, p\}$. Hence, the LCP problem in (A.2) is equivalent to the following p LCP subproblems which can be solved by a back substitution type process. Namely,

$$r_p = A_{pp} z_p + q_p, \quad r_i = A_{ii} z_i + \left(\sum_{j=i+1}^{p} (A_{ij} z_j) + q_i \right), \quad i = p-1, p-2, \cdots, 2, 1,$$

where the vector $\sum_{j=i+1}^{p} (A_{ij} z_j) + q_i$ plays the role of the known vector.

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